6.3   Vectors in the Plane

What you should learn

- Represent vectors as directed line segments.
- Write the component forms of vectors.
- Perform basic vector operations and represent them graphically.
- Write vectors as linear combinations of unit vectors.
- Find the direction angles of vectors.
- Use vectors to model and solve real-life problems.

Why you should learn it

You can use vectors to model and solve real-life problems involving magnitude and direction. For instance, in Exercise 84 on page 459, you can use vectors to determine the true direction of a commercial jet.

Bill Bachman/Photo Researchers, Inc.

Introduction

Quantities such as force and velocity involve both *magnitude* and *direction* and cannot be completely characterized by a single real number. To represent such a quantity, you can use a **directed line segment,** as shown in Figure 6.15. The directed line segment $\overrightarrow{PQ}$ has **initial point** P and **terminal point** Q. Its **magnitude** (or length) is denoted by $\|\overrightarrow{PQ}\|$ and can be found using the Distance Formula.
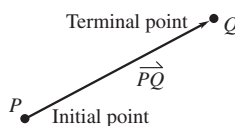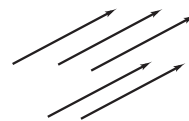

FIGURE 6.15   FIGURE 6.16

Two directed line segments that have the same magnitude and direction are equivalent. For example, the directed line segments in Figure 6.16 are all equivalent. The set of all directed line segments that are equivalent to the directed line segment $\overrightarrow{PQ}$ is a **vector v in the plane,** written $\mathbf{v} = \overrightarrow{PQ}$. Vectors are denoted by lowercase, boldface letters such as $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Example 1   **Vector Representation by Directed Line Segments**

Let $\mathbf{u}$ be represented by the directed line segment from $P = (0, 0)$ to $Q = (3, 2)$, and let $\mathbf{v}$ be represented by the directed line segment from $R = (1, 2)$ to $S = (4, 4)$, as shown in Figure 6.17. Show that $\mathbf{u} = \mathbf{v}$.
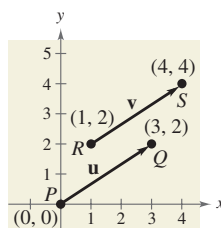

FIGURE 6.17

Solution

From the Distance Formula, it follows that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the *same magnitude*.

$$\|\overrightarrow{PQ}\| = \sqrt{(3 - 0)^2 + (2 - 0)^2} = \sqrt{13}$$

$$\|\overrightarrow{RS}\| = \sqrt{(4 - 1)^2 + (4 - 2)^2} = \sqrt{13}$$

Moreover, both line segments have the *same direction* because they are both directed toward the upper right on lines having a slope of $\frac{2}{3}$. So, $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, and it follows that $\mathbf{u} = \mathbf{v}$.

✔CHECKPOINT   Now try Exercise 1.

Component Form of a Vector

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments. This representative of the vector **v** is in **standard position.**

A vector whose initial point is the origin $(0, 0)$ can be uniquely represented by the coordinates of its terminal point (v_1, v_2). This is the **component form of a vector v,** written as

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

The coordinates v_1 and v_2 are the *components* of **v**. If both the initial point and the terminal point lie at the origin, **v** is the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

> Help students see that $\mathbf{v} = \langle 1, 3 \rangle$ can be thought of as a vector with initial point $(0, 0)$ and terminal point $(1, 3)$, as well as a vector with initial point $(0, -1)$ and terminal point $(1, 2)$, and so on.

Component Form of a Vector

The component form of the vector with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is given by

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

The **magnitude** (or length) of **v** is given by

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

If $\|\mathbf{v}\| = 1$, **v** is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if **v** is the zero vector **0**.

Technology

You can graph vectors with a graphing utility by graphing directed line segments. Consult the user's guide for your graphing utility for specific instructions.
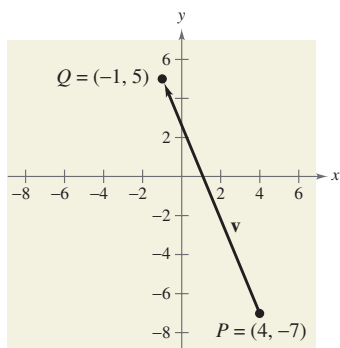
Two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are *equal* if and only if $u_1 = v_1$ and $u_2 = v_2$. For instance, in Example 1, the vector **u** from $P = (0, 0)$ to $Q = (3, 2)$ is

$$\mathbf{u} = \overrightarrow{PQ} = \langle 3 - 0, 2 - 0 \rangle = \langle 3, 2 \rangle$$

and the vector **v** from $R = (1, 2)$ to $S = (4, 4)$ is

$$\mathbf{v} = \overrightarrow{RS} = \langle 4 - 1, 4 - 2 \rangle = \langle 3, 2 \rangle.$$

Example 2 Finding the Component Form of a Vector

Find the component form and magnitude of the vector **v** that has initial point $(4, -7)$ and terminal point $(-1, 5)$.

Solution

Let $P = (4, -7) = (p_1, p_2)$ and let $Q = (-1, 5) = (q_1, q_2)$, as shown in Figure 6.18. Then, the components of $\mathbf{v} = \langle v_1, v_2 \rangle$ are

$$v_1 = q_1 - p_1 = -1 - 4 = -5$$
$$v_2 = q_2 - p_2 = 5 - (-7) = 12.$$

So, $\mathbf{v} = \langle -5, 12 \rangle$ and the magnitude of **v** is

$$\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2}$$
$$= \sqrt{169} = 13.$$

FIGURE 6.18

✓CHECKPOINT   Now try Exercise 9.
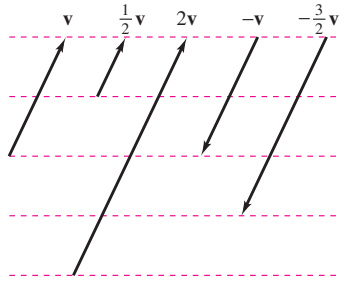
Vector Operations

The two basic vector operations are **scalar multiplication** and **vector addition.** In operations with vectors, numbers are usually referred to as **scalars.** In this text, scalars will always be real numbers. Geometrically, the product of a vector **v** and a scalar k is the vector that is $|k|$ times as long as **v**. If k is positive, k**v** has the same direction as **v**, and if k is negative, k**v** has the direction opposite that of **v**, as shown in Figure 6.19.

To add two vectors geometrically, position them (without changing their lengths or directions) so that the initial point of one coincides with the terminal point of the other. The sum **u** + **v** is formed by joining the initial point of the second vector **v** with the terminal point of the first vector **u**, as shown in Figure 6.20. This technique is called the **parallelogram law** for vector addition because the vector **u** + **v**, often called the **resultant** of vector addition, is the diagonal of a parallelogram having **u** and **v** as its adjacent sides.
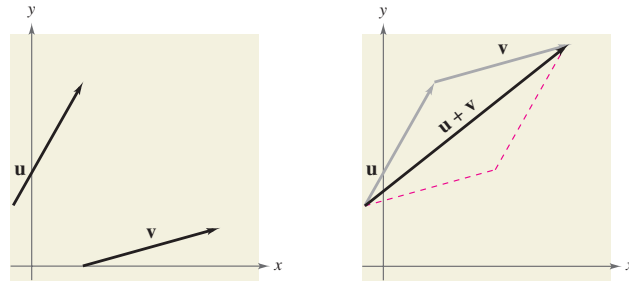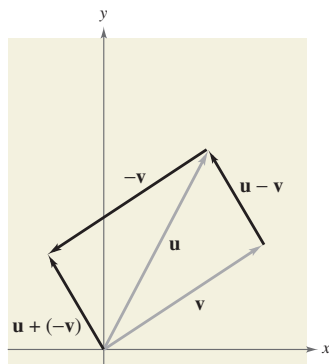

FIGURE 6.20

The graphical representation of the difference of two vectors may not be obvious to your students. You may want to go over this carefully.

u − **v** = **u** + (−**v**)

FIGURE 6.21

Definitions of Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a scalar (a real number). Then the *sum* of **u** and **v** is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \qquad \text{Sum}$$

and the *scalar multiple* of k times **u** is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle. \qquad \text{Scalar multiple}$$

The **negative** of $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$-\mathbf{v} = (-1)\mathbf{v}$$

$$= \langle -v_1, -v_2 \rangle \qquad \text{Negative}$$

and the **difference** of **u** and **v** is

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) \qquad \text{Add } (-\mathbf{v}). \text{ See Figure 8.21.}$$

$$= \langle u_1 - v_1, u_2 - v_2 \rangle. \qquad \text{Difference}$$

To represent **u** − **v** geometrically, you can use directed line segments with the *same* initial point. The difference **u** − **v** is the vector from the terminal point of **v** to the terminal point of **u**, which is equal to **u** + (−**v**), as shown in Figure 6.21.

The component definitions of vector addition and scalar multiplication are illustrated in Example 3. In this example, notice that each of the vector operations can be interpreted geometrically.

Example 3 **Vector Operations**

Let $\mathbf{v} = \langle -2, 5 \rangle$ and $\mathbf{w} = \langle 3, 4 \rangle$, and find each of the following vectors.

a. $2\mathbf{v}$ **b.** $\mathbf{w} - \mathbf{v}$ **c.** $\mathbf{v} + 2\mathbf{w}$

Solution

a. Because $\mathbf{v} = \langle -2, 5 \rangle$, you have

$$2\mathbf{v} = 2\langle -2, 5 \rangle$$
$$= \langle 2(-2), 2(5) \rangle$$
$$= \langle -4, 10 \rangle.$$

A sketch of $2\mathbf{v}$ is shown in Figure 6.22.

b. The difference of $\mathbf{w}$ and $\mathbf{v}$ is

$$\mathbf{w} - \mathbf{v} = \langle 3 - (-2), 4 - 5 \rangle$$
$$= \langle 5, -1 \rangle.$$

A sketch of $\mathbf{w} - \mathbf{v}$ is shown in Figure 6.23. Note that the figure shows the vector difference $\mathbf{w} - \mathbf{v}$ as the sum $\mathbf{w} + (-\mathbf{v})$.

c. The sum of $\mathbf{v}$ and $2\mathbf{w}$ is

$$\mathbf{v} + 2\mathbf{w} = \langle -2, 5 \rangle + 2\langle 3, 4 \rangle$$
$$= \langle -2, 5 \rangle + \langle 2(3), 2(4) \rangle$$
$$= \langle -2, 5 \rangle + \langle 6, 8 \rangle$$
$$= \langle -2 + 6, 5 + 8 \rangle$$
$$= \langle 4, 13 \rangle.$$

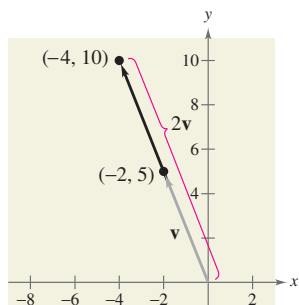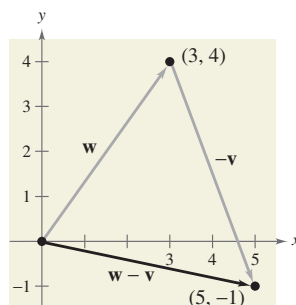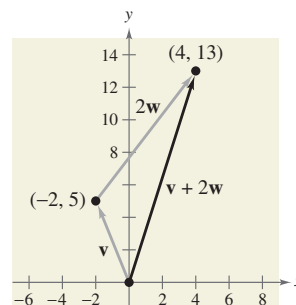A sketch of $\mathbf{v} + 2\mathbf{w}$ is shown in Figure 6.24.

FIGURE 6.22

FIGURE 6.23

FIGURE 6.24

CHECKPOINT Now try Exercise 21.

Vector addition and scalar multiplication share many of the properties of ordinary arithmetic.

> ### Properties of Vector Addition and Scalar Multiplication
>
> Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and let c and d be scalars. Then the following properties are true.
>
> 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
>
> 3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ 4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
>
> 5. $c(d\mathbf{u}) = (cd)\mathbf{u}$ 6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
>
> 7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ 8. $1(\mathbf{u}) = \mathbf{u}, 0(\mathbf{u}) = \mathbf{0}$
>
> 9. $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$

Property 9 can be stated as follows: the magnitude of the vector $c\mathbf{v}$ is the absolute value of c times the magnitude of $\mathbf{v}$.

Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector $\mathbf{v}$. To do this, you can divide $\mathbf{v}$ by its magnitude to obtain

$$\mathbf{u} = \text{unit vector} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}. \qquad \text{Unit vector in direction of } \mathbf{v}$$

Note that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. The vector $\mathbf{u}$ has a magnitude of 1 and the same direction as $\mathbf{v}$. The vector $\mathbf{u}$ is called a **unit vector in the direction of** $\mathbf{v}$.

Example 4   Finding a Unit Vector

Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5\rangle$ and verify that the result has a magnitude of 1.

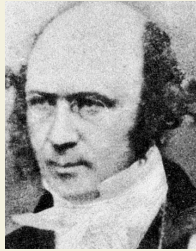#### Solution

The unit vector in the direction of $\mathbf{v}$ is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5\rangle}{\sqrt{(-2)^2 + (5)^2}}$$

$$= \frac{1}{\sqrt{29}}\langle -2, 5\rangle$$

$$= \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right\rangle.$$

This vector has a magnitude of 1 because

$$\sqrt{\left(\frac{-2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{\frac{4}{29} + \frac{25}{29}} = \sqrt{\frac{29}{29}} = 1.$$

✔CHECKPOINT   Now try Exercise 31.

The Granger Collection

Historical Note
William Rowan Hamilton (1805–1865), an Irish mathematician, did some of the earliest work with vectors. Hamilton spent many years developing a system of vector-like quantities called quaternions. Although Hamilton was convinced of the benefits of quaternions, the operations he defined did not produce good models for physical phenomena. It wasn't until the latter half of the nineteenth century that the Scottish physicist James Maxwell (1831–1879) restructured Hamilton's quaternions in a form useful for representing physical quantities such as force, velocity, and acceleration.
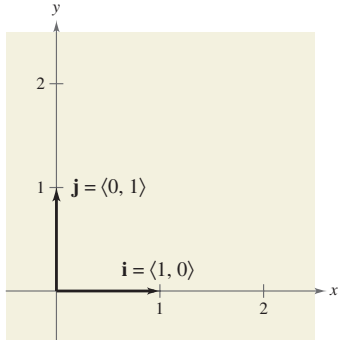
FIGURE 6.25

The unit vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are called the **standard unit vectors** and are denoted by

$$\mathbf{i} = \langle 1, 0 \rangle \qquad \text{and} \qquad \mathbf{j} = \langle 0, 1 \rangle$$

as shown in Figure 6.25. (Note that the lowercase letter $\mathbf{i}$ is written in boldface to distinguish it from the imaginary number $i = \sqrt{-1}$.) These vectors can be used to represent any vector $\mathbf{v} = \langle v_1, v_2 \rangle$, as follows.

$$\begin{aligned} \mathbf{v} &= \langle v_1, v_2 \rangle \\ &= v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle \\ &= v_1 \mathbf{i} + v_2 \mathbf{j} \end{aligned}$$

The scalars v_1 and v_2 are called the **horizontal** and **vertical components of v,** respectively. The vector sum

$$v_1 \mathbf{i} + v_2 \mathbf{j}$$

is called a **linear combination** of the vectors $\mathbf{i}$ and $\mathbf{j}$. Any vector in the plane can be written as a linear combination of the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.
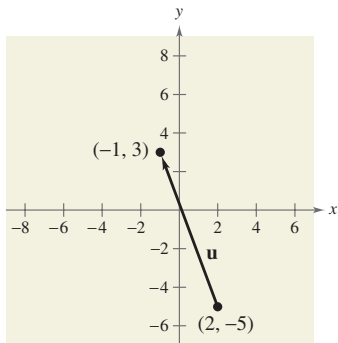

FIGURE 6.26

Example 5   **Writing a Linear Combination of Unit Vectors**

Let $\mathbf{u}$ be the vector with initial point $(2, -5)$ and terminal point $(-1, 3)$. Write $\mathbf{u}$ as a linear combination of the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Solution

Begin by writing the component form of the vector $\mathbf{u}$.

$$\begin{aligned} \mathbf{u} &= \langle -1 - 2, 3 - (-5) \rangle \\ &= \langle -3, 8 \rangle \\ &= -3\mathbf{i} + 8\mathbf{j} \end{aligned}$$

This result is shown graphically in Figure 6.26.

✓CHECKPOINT   Now try Exercise 43.

Example 6   **Vector Operations**

Let $\mathbf{u} = -3\mathbf{i} + 8\mathbf{j}$ and let $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Find $2\mathbf{u} - 3\mathbf{v}$.

Solution

You could solve this problem by converting $\mathbf{u}$ and $\mathbf{v}$ to component form. This, however, is not necessary. It is just as easy to perform the operations in unit vector form.

$$\begin{aligned} 2\mathbf{u} - 3\mathbf{v} &= 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j}) \\ &= -6\mathbf{i} + 16\mathbf{j} - 6\mathbf{i} + 3\mathbf{j} \\ &= -12\mathbf{i} + 19\mathbf{j} \end{aligned}$$
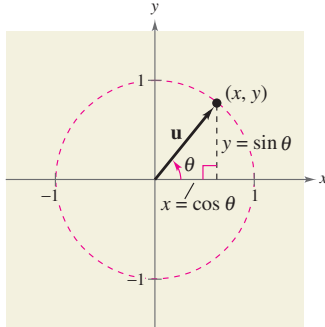
✓CHECKPOINT   Now try Exercise 49.

Direction Angles

FIGURE 6.27 $\|\mathbf{u}\| = 1$

If $\mathbf{u}$ is a *unit vector* such that θ is the angle (measured counterclockwise) from the positive x-axis to $\mathbf{u}$, the terminal point of $\mathbf{u}$ lies on the unit circle and you have

$$\mathbf{u} = \langle x, y \rangle = \langle \cos\theta, \sin\theta \rangle = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$$

as shown in Figure 6.27. The angle θ is the **direction angle** of the vector $\mathbf{u}$.

Suppose that $\mathbf{u}$ is a unit vector with direction angle θ. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is any vector that makes an angle θ with the positive x-axis, it has the same direction as $\mathbf{u}$ and you can write

$$\mathbf{v} = \|\mathbf{v}\| \langle \cos\theta, \sin\theta \rangle$$
$$= \|\mathbf{v}\| (\cos\theta)\mathbf{i} + \|\mathbf{v}\| (\sin\theta)\mathbf{j}.$$

Because $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\| (\cos\theta)\mathbf{i} + \|\mathbf{v}\| (\sin\theta)\mathbf{j}$, it follows that the direction angle θ for $\mathbf{v}$ is determined from

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \text{Quotient identity}$$

$$= \frac{\|\mathbf{v}\| \sin\theta}{\|\mathbf{v}\| \cos\theta} \qquad \text{Multiply numerator and denominator by } \|\mathbf{v}\|.$$

$$= \frac{b}{a}. \qquad \text{Simplify.}$$

Example 7   Finding Direction Angles of Vectors

Find the direction angle of each vector.

a. $\mathbf{u} = 3\mathbf{i} + 3\mathbf{j}$

b. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

Solution

a. The direction angle is

$$\tan\theta = \frac{b}{a} = \frac{3}{3} = 1.$$

FIGURE 6.28

So, $\theta = 45°$, as shown in Figure 6.28.

b. The direction angle is

$$\tan\theta = \frac{b}{a} = \frac{-4}{3}.$$

Moreover, because $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ lies in Quadrant IV, θ lies in Quadrant IV and its reference angle is

$$\theta = \left| \arctan\left(-\frac{4}{3}\right) \right| \approx |-53.13°| = 53.13°.$$

FIGURE 6.29

So, it follows that $\theta \approx 360° - 53.13° = 306.87°$, as shown in Figure 6.29.

✓CHECKPOINT   Now try Exercise 55.
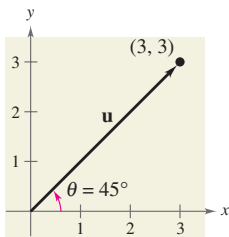
Applications of Vectors
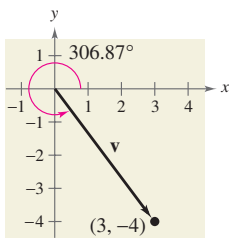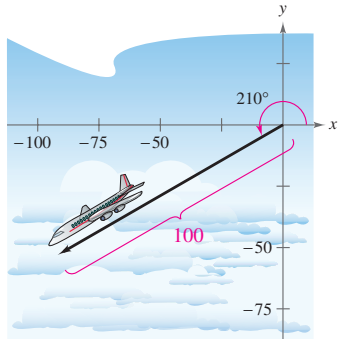

FIGURE 6.30

| Example 8 | **Finding the Component Form of a Vector** |

Find the component form of the vector that represents the velocity of an airplane descending at a speed of 100 miles per hour at an angle $30°$ below the horizontal, as shown in Figure 6.30.

Solution

The velocity vector $\mathbf{v}$ has a magnitude of 100 and a direction angle of $\theta = 210°$.

$$\mathbf{v} = \|\mathbf{v}\| (\cos \theta)\mathbf{i} + \|\mathbf{v}\| (\sin \theta)\mathbf{j}$$

$$= 100(\cos 210°)\mathbf{i} + 100(\sin 210°)\mathbf{j}$$

$$= 100\left(-\frac{\sqrt{3}}{2}\right)\mathbf{i} + 100\left(-\frac{1}{2}\right)\mathbf{j}$$

$$= -50\sqrt{3}\,\mathbf{i} - 50\mathbf{j}$$

$$= \langle -50\sqrt{3}, -50 \rangle$$

You can check that $\mathbf{v}$ has a magnitude of 100, as follows.

$$\|\mathbf{v}\| = \sqrt{\left(-50\sqrt{3}\right)^2 + (-50)^2}$$

$$= \sqrt{7500 + 2500}$$

$$= \sqrt{10,000} = 100$$

✔CHECKPOINT   Now try Exercise 77.

| Example 9 | **Using Vectors to Determine Weight** |

A force of 600 pounds is required to pull a boat and trailer up a ramp inclined at $15°$ from the horizontal. Find the combined weight of the boat and trailer.

Solution

Based on Figure 6.31, you can make the following observations.

$$\|\overrightarrow{BA}\| = \text{force of gravity} = \text{combined weight of boat and trailer}$$

$$\|\overrightarrow{BC}\| = \text{force against ramp}$$

$$\|\overrightarrow{AC}\| = \text{force required to move boat up ramp} = 600 \text{ pounds}$$

FIGURE 6.31

By construction, triangles BWD and ABC are similar. So, angle ABC is $15°$, and so in triangle ABC you have

$$\sin 15° = \frac{\|\overrightarrow{AC}\|}{\|\overrightarrow{BA}\|} = \frac{600}{\|\overrightarrow{BA}\|}$$

$$\|\overrightarrow{BA}\| = \frac{600}{\sin 15°} \approx 2318.$$

Consequently, the combined weight is approximately 2318 pounds. (In Figure 6.31, note that $\overrightarrow{AC}$ is parallel to the ramp.)
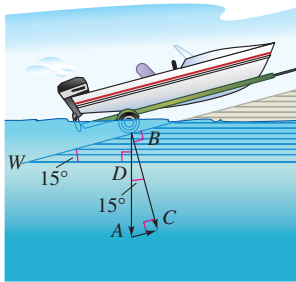
✔CHECKPOINT   Now try Exercise 81.

Example 10   **Using Vectors to Find Speed and Direction**

An airplane is traveling at a speed of 500 miles per hour with a bearing of 330° at a fixed altitude with a negligible wind velocity as shown in Figure 6.32(a). When the airplane reaches a certain point, it encounters a wind with a velocity of 70 miles per hour in the direction N 45° E, as shown in Figure 6.32(b). What are the resultant speed and direction of the airplane?
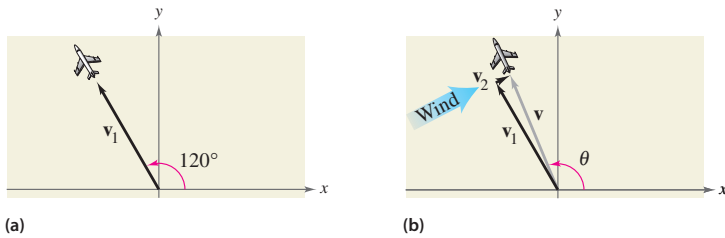
(a)

(b)

FIGURE 6.32

Solution

Using Figure 6.32, the velocity of the airplane (alone) is

$$\mathbf{v}_1 = 500\langle \cos 120°, \sin 120° \rangle$$

$$= \langle -250, 250\sqrt{3} \rangle$$

and the velocity of the wind is

$$\mathbf{v}_2 = 70\langle \cos 45°, \sin 45° \rangle$$

$$= \langle 35\sqrt{2}, 35\sqrt{2} \rangle.$$

So, the velocity of the airplane (in the wind) is

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

$$= \langle -250 + 35\sqrt{2}, 250\sqrt{3} + 35\sqrt{2} \rangle$$

$$\approx \langle -200.5, 482.5 \rangle$$

and the resultant speed of the airplane is

$$\|\mathbf{v}\| = \sqrt{(-200.5)^2 + (482.5)^2}$$

$$\approx 522.5 \text{ miles per hour.}$$

Finally, if θ is the direction angle of the flight path, you have

$$\tan \theta = \frac{482.5}{-200.5}$$

$$\approx -2.4065$$

which implies that

$$\theta \approx 180° + \arctan(-2.4065) \approx 180° - 67.4° = 112.6°.$$

So, the true direction of the airplane is 337.4°.

✔CHECKPOINT   Now try Exercise 83.
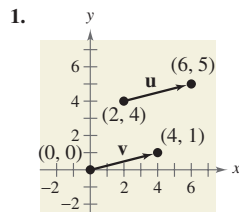
6.3 | Exercises

VOCABULARY CHECK: Fill in the blanks.

1. A _____ _____ _____ can be used to represent a quantity that involves both magnitude and direction.
2. The directed line segment $\overrightarrow{PQ}$ has _____ point P and _____ point Q.
3. The _____ of the directed line segment $\overrightarrow{PQ}$ is denoted by $\|PQ\|$.
4. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a _____ **v** in the plane.
5. The directed line segment whose initial point is the origin is said to be in _____ _____ .
6. A vector that has a magnitude of 1 is called a _____ _____ .
7. The two basic vector operations are scalar _____ and vector _____ .
8. The vector **u** + **v** is called the _____ of vector addition.
9. The vector sum $v_1\mathbf{i} + v_2\mathbf{j}$ is called a _____ _____ of the vectors **i** and **j**, and the scalars v_1 and v_2 are called the _____ and _____ components of **v**, respectively.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**
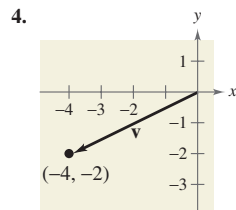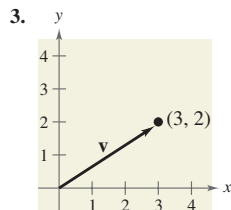
In Exercises 1 and 2, show that **u** = **v**.

1.
2.
7.
8.

In Exercises 3–14, find the component form and the magnitude of the vector **v**.

3.
4.
5.
6.
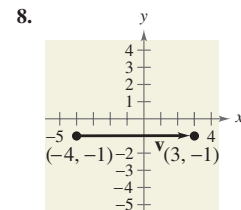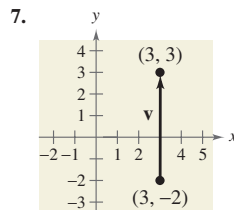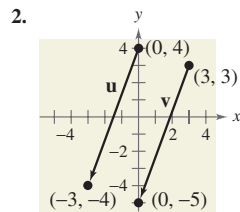
	Initial Point	Terminal Point
9.	$(-1, 5)$	$(15, 12)$
10.	$(1, 11)$	$(9, 3)$
11.	$(-3, -5)$	$(5, 1)$
12.	$(-3, 11)$	$(9, 40)$
13.	$(1, 3)$	$(-8, -9)$
14.	$(-2, 7)$	$(5, -17)$

In Exercises 15–20, use the figure to sketch a graph of the specified vector. To print an enlarged copy of the graph, go to the website, *www.mathgraphs.com.*
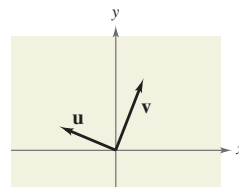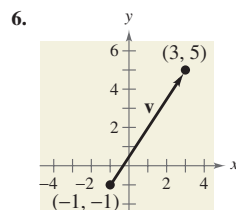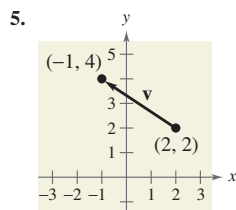
15. $-\mathbf{v}$
16. $5\mathbf{v}$
17. $\mathbf{u} + \mathbf{v}$
18. $\mathbf{u} - \mathbf{v}$
19. $\mathbf{u} + 2\mathbf{v}$
20. $\mathbf{v} - \frac{1}{2}\mathbf{u}$

In Exercises 21–28, find (a) $\mathbf{u} + \mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, and (c) $2\mathbf{u} - 3\mathbf{v}$. Then sketch the resultant vector.

21. $\mathbf{u} = \langle 2, 1 \rangle$, $\mathbf{v} = \langle 1, 3 \rangle$

22. $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle 4, 0 \rangle$

23. $\mathbf{u} = \langle -5, 3 \rangle$, $\mathbf{v} = \langle 0, 0 \rangle$

24. $\mathbf{u} = \langle 0, 0 \rangle$, $\mathbf{v} = \langle 2, 1 \rangle$

25. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

26. $\mathbf{u} = -2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = -\mathbf{i} + 2\mathbf{j}$

27. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = \mathbf{j}$

28. $\mathbf{u} = 3\mathbf{j}$, $\mathbf{v} = 2\mathbf{i}$

In Exercises 29–38, find a unit vector in the direction of the given vector.

29. $\mathbf{u} = \langle 3, 0 \rangle$

30. $\mathbf{u} = \langle 0, -2 \rangle$

31. $\mathbf{v} = \langle -2, 2 \rangle$

32. $\mathbf{v} = \langle 5, -12 \rangle$

33. $\mathbf{v} = 6\mathbf{i} - 2\mathbf{j}$

34. $\mathbf{v} = \mathbf{i} + \mathbf{j}$

35. $\mathbf{w} = 4\mathbf{j}$

36. $\mathbf{w} = -6\mathbf{i}$

37. $\mathbf{w} = \mathbf{i} - 2\mathbf{j}$

38. $\mathbf{w} = 7\mathbf{j} - 3\mathbf{i}$

In Exercises 39–42, find the vector $\mathbf{v}$ with the given magnitude and the same direction as $\mathbf{u}$.

Magnitude	Direction
39. $\|\mathbf{v}\| = 5$	$\mathbf{u} = \langle 3, 3 \rangle$
40. $\|\mathbf{v}\| = 6$	$\mathbf{u} = \langle -3, 3 \rangle$
41. $\|\mathbf{v}\| = 9$	$\mathbf{u} = \langle 2, 5 \rangle$
42. $\|\mathbf{v}\| = 10$	$\mathbf{u} = \langle -10, 0 \rangle$

In Exercises 43–46, the initial and terminal points of a vector are given. Write a linear combination of the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Initial Point	Terminal Point
43. $(-3, 1)$	$(4, 5)$
44. $(0, -2)$	$(3, 6)$
45. $(-1, -5)$	$(2, 3)$
46. $(-6, 4)$	$(0, 1)$

In Exercises 47–52, find the component form of $\mathbf{v}$ and sketch the specified vector operations geometrically, where $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j}$.

47. $\mathbf{v} = \frac{3}{2}\mathbf{u}$

48. $\mathbf{v} = \frac{3}{4}\mathbf{w}$

49. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$

50. $\mathbf{v} = -\mathbf{u} + \mathbf{w}$

51. $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$

52. $\mathbf{v} = \mathbf{u} - 2\mathbf{w}$

In Exercises 53–56, find the magnitude and direction angle of the vector $\mathbf{v}$.

53. $\mathbf{v} = 3(\cos 60°\mathbf{i} + \sin 60°\mathbf{j})$

54. $\mathbf{v} = 8(\cos 135°\mathbf{i} + \sin 135°\mathbf{j})$

55. $\mathbf{v} = 6\mathbf{i} - 6\mathbf{j}$

56. $\mathbf{v} = -5\mathbf{i} + 4\mathbf{j}$

In Exercises 57–64, find the component form of $\mathbf{v}$ given its magnitude and the angle it makes with the positive x-axis. Sketch $\mathbf{v}$.

Magnitude	Angle
57. $\|\mathbf{v}\| = 3$	$\theta = 0°$
58. $\|\mathbf{v}\| = 1$	$\theta = 45°$
59. $\|\mathbf{v}\| = \frac{7}{2}$	$\theta = 150°$
60. $\|\mathbf{v}\| = \frac{5}{2}$	$\theta = 45°$
61. $\|\mathbf{v}\| = 3\sqrt{2}$	$\theta = 150°$
62. $\|\mathbf{v}\| = 4\sqrt{3}$	$\theta = 90°$
63. $\|\mathbf{v}\| = 2$	$\mathbf{v}$ in the direction $\mathbf{i} + 3\mathbf{j}$
64. $\|\mathbf{v}\| = 3$	$\mathbf{v}$ in the direction $3\mathbf{i} + 4\mathbf{j}$

In Exercises 65–68, find the component form of the sum of $\mathbf{u}$ and $\mathbf{v}$ with direction angles θ_u and θ_v.

Magnitude	Angle
65. $\|\mathbf{u}\| = 5$	$\theta_{\mathbf{u}} = 0°$
$\|\mathbf{v}\| = 5$	$\theta_{\mathbf{v}} = 90°$
66. $\|\mathbf{u}\| = 4$	$\theta_{\mathbf{u}} = 60°$
$\|\mathbf{v}\| = 4$	$\theta_{\mathbf{v}} = 90°$
67. $\|\mathbf{u}\| = 20$	$\theta_{\mathbf{u}} = 45°$
$\|\mathbf{v}\| = 50$	$\theta_{\mathbf{v}} = 180°$
68. $\|\mathbf{u}\| = 50$	$\theta_{\mathbf{u}} = 30°$
$\|\mathbf{v}\| = 30$	$\theta_{\mathbf{v}} = 110°$

In Exercises 69 and 70, use the Law of Cosines to find the angle α between the vectors. (Assume $0° \leq \alpha \leq 180°$.)

69. $\mathbf{v} = \mathbf{i} + \mathbf{j}$, $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j}$

70. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$

Resultant Force   In Exercises 71 and 72, find the angle between the forces given the magnitude of their resultant. (*Hint*: Write force 1 as a vector in the direction of the positive x-axis and force 2 as a vector at an angle θ with the positive x-axis.)
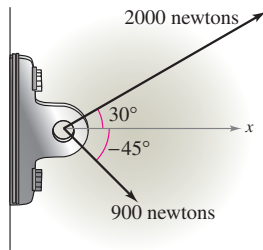
Force 1	Force 2	Resultant Force
71. 45 pounds	60 pounds	90 pounds
72. 3000 pounds	1000 pounds	3750 pounds

73. *Resultant Force* Forces with magnitudes of 125 newtons and 300 newtons act on a hook (see figure). The angle between the two forces is 45°. Find the direction and magnitude of the resultant of these forces.
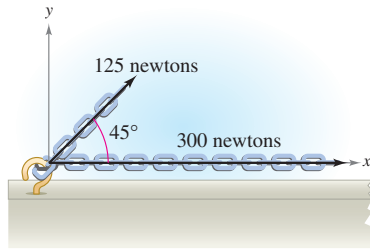

74. *Resultant Force* Forces with magnitudes of 2000 newtons and 900 newtons act on a machine part at angles of 30° and −45°, respectively, with the x-axis (see figure). Find the direction and magnitude of the resultant of these forces.

75. *Resultant Force* Three forces with magnitudes of 75 pounds, 100 pounds, and 125 pounds act on an object at angles of 30°, 45°, and 120°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant of these forces.

76. *Resultant Force* Three forces with magnitudes of 70 pounds, 40 pounds, and 60 pounds act on an object at angles of −30°, 445°, and 135°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant of these forces.

77. *Velocity* A ball is thrown with an initial velocity of 70 feet per second, at an angle of 35° with the horizontal (see figure). Find the vertical and horizontal components of the velocity.
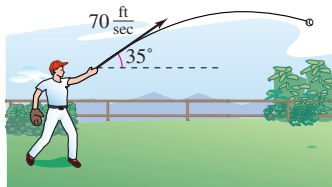

78. *Velocity* A gun with a muzzle velocity of 1200 feet per second is fired at an angle of 6° with the horizontal. Find the vertical and horizontal components of the velocity.

Cable Tension **In Exercises 79 and 80, use the figure to determine the tension in each cable supporting the load.**

79.

80.
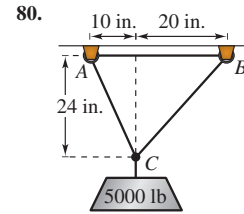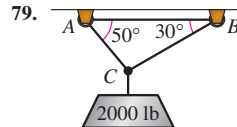
81. *Tow Line Tension* A loaded barge is being towed by two tugboats, and the magnitude of the resultant is 6000 pounds directed along the axis of the barge (see figure). Find the tension in the tow lines if they each make an 18° angle with the axis of the barge.
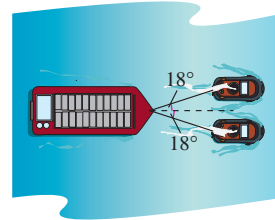
82. *Rope Tension* To carry a 100-pound cylindrical weight, two people lift on the ends of short ropes that are tied to an eyelet on the top center of the cylinder. Each rope makes a 20° angle with the vertical. Draw a figure that gives a visual representation of the problem, and find the tension in the ropes.

83. *Navigation* An airplane is flying in the direction of 148°, with an airspeed of 875 kilometers per hour. Because of the wind, its groundspeed and direction are 800 kilometers per hour and 140°, respectively (see figure). Find the direction and speed of the wind.
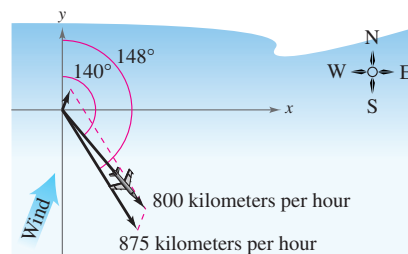
Model It

84. *Navigation*   A commercial jet is flying from Miami to Seattle. The jet's velocity with respect to the air is 580 miles per hour, and its bearing is 332°. The wind, at the altitude of the plane, is blowing from the southwest with a velocity of 60 miles per hour.

(a) Draw a figure that gives a visual representation of the problem.

(b) Write the velocity of the wind as a vector in component form.

(c) Write the velocity of the jet relative to the air in component form.

(d) What is the speed of the jet with respect to the ground?

(e) What is the true direction of the jet?

85. *Work*   A heavy implement is pulled 30 feet across a floor, using a force of 100 pounds. The force is exerted at an angle of 50° above the horizontal (see figure). Find the work done. (Use the formula for work, $W = FD$, where F is the component of the force in the direction of motion and D is the distance.)
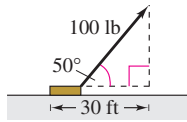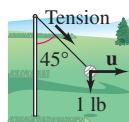
FIGURE FOR 85   FIGURE FOR 86

86. *Rope Tension*   A tetherball weighing 1 pound is pulled outward from the pole by a horizontal force **u** until the rope makes a 45° angle with the pole (see figure). Determine the resulting tension in the rope and the magnitude of **u**.

Synthesis

True or False?   In Exercises 87 and 88, decide whether the statement is true or false. Justify your answer.

87. If **u** and **v** have the same magnitude and direction, then **u** = **v**.

88. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $a^2 + b^2 = 1$.

89. *Think About It*   Consider two forces of equal magnitude acting on a point.

(a) If the magnitude of the resultant is the sum of the magnitudes of the two forces, make a conjecture about the angle between the forces.

(b) If the resultant of the forces is **0**, make a conjecture about the angle between the forces.

(c) Can the magnitude of the resultant be greater than the sum of the magnitudes of the two forces? Explain.

90. *Graphical Reasoning*   Consider two forces

$$\mathbf{F}_1 = \langle 10, 0 \rangle \text{ and } \mathbf{F}_2 = 5\langle \cos \theta, \sin \theta \rangle.$$

(a) Find $\|\mathbf{F}_1 + \mathbf{F}_2\|$ as a function of θ.

(b) Use a graphing utility to graph the function in part (a) for $0 \le \theta < 2\pi$.

(c) Use the graph in part (b) to determine the range of the function. What is its maximum, and for what value of θ does it occur? What is its minimum, and for what value of θ does it occur?

(d) Explain why the magnitude of the resultant is never 0.

91. *Proof*   Prove that $(\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$ is a unit vector for any value of θ.

92. *Technology*   Write a program for your graphing utility that graphs two vectors and their difference given the vectors in component form.

In Exercises 93 and 94, use the program in Exercise 92 to find the difference of the vectors shown in the figure.

93.

94.
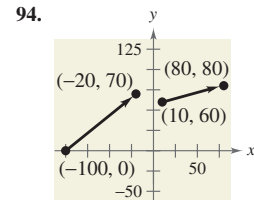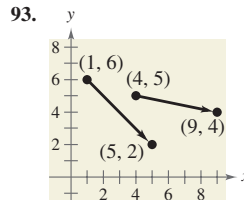
Skills Review

In Exercises 95–98, use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ, where $0 < \theta < \pi/2$.

95. $\sqrt{x^2 - 64}, \quad x = 8 \sec \theta$

96. $\sqrt{64 - x^2}, \quad x = 8 \sin \theta$

97. $\sqrt{x^2 + 36}, \quad x = 6 \tan \theta$

98. $\sqrt{(x^2 - 25)^3}, \quad x = 5 \sec \theta$

In Exercises 99–102, solve the equation.

99. $\cos x(\cos x + 1) = 0$

100. $\sin x\left(2 \sin x + \sqrt{2}\right) = 0$

101. $3 \sec x \sin x - 2\sqrt{3} \sin x = 0$

102. $\cos x \csc x + \cos x\sqrt{2} = 0$