12.4 EXERCISES

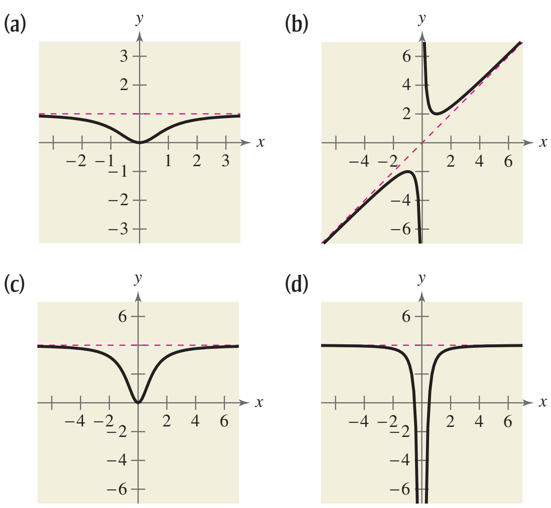
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blanks.

1. A ________ at ________ can be used to solve the area problem in calculus.
2. When evaluating limits at infinity for complicated rational functions, you can divide the numerator and denominator by the ________ term in the denominator.
3. A sequence that has a limit is said to ________.
4. A sequence that does not have a limit is said to ________.

SKILLS AND APPLICATIONS

In Exercises 5–8, match the function with its graph, using horizontal asymptotes as aids. [The graphs are labeled (a), (b), (c), and (d).]



5. $f(x) = \dfrac{4x^2}{x^2 + 1}$
6. $f(x) = \dfrac{x^2}{x^2 + 1}$
7. $f(x) = 4 - \dfrac{1}{x^2}$
8. $f(x) = x + \dfrac{1}{x}$

In Exercises 9–28, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

9. $\lim\limits_{x\to\infty}\left(\dfrac{3}{x^2} + 1\right)$
10. $\lim\limits_{x\to\infty}\left(\dfrac{4}{3x} - 5\right)$
11. $\lim\limits_{x\to\infty}\left(\dfrac{1 - x}{1 + x}\right)$
12. $\lim\limits_{x\to\infty}\left(\dfrac{1 + 5x}{1 - 4x}\right)$
13. $\lim\limits_{x\to-\infty}\dfrac{4x - 3}{2x + 1}$
14. $\lim\limits_{x\to\infty}\dfrac{1 - 2x}{x + 2}$
15. $\lim\limits_{x\to-\infty}\dfrac{3x^2 - 4}{1 - x^2}$
16. $\lim\limits_{x\to-\infty}\dfrac{3x^2 + 1}{4x^2 - 5}$
17. $\lim\limits_{t\to\infty}\dfrac{t^2}{t + 3}$
18. $\lim\limits_{y\to\infty}\dfrac{4y^4}{y^2 + 3}$
19. $\lim\limits_{t\to\infty}\dfrac{4t^2 - 2t + 1}{-3t^2 + 2t + 2}$
20. $\lim\limits_{x\to-\infty}\dfrac{2x^2 - 5x - 12}{1 - 6x - 8x^2}$
21. $\lim\limits_{x\to-\infty}\dfrac{-(x^2 + 3)}{(2 - x)^2}$
22. $\lim\limits_{x\to\infty}\dfrac{2x^2 - 6}{(x - 1)^2}$
23. $\lim\limits_{x\to\infty}\dfrac{5x^3 + 1}{10x^3 - 3x^2 + 7}$
24. $\lim\limits_{x\to-\infty}\left(\dfrac{1}{2}x - \dfrac{4}{x^2}\right)$
25. $\lim\limits_{x\to-\infty}\left[\dfrac{x}{(x + 1)^2} - 4\right]$
26. $\lim\limits_{x\to\infty}\left[7 + \dfrac{2x^2}{(x + 3)^2}\right]$
27. $\lim\limits_{t\to\infty}\left(\dfrac{1}{3t^2} - \dfrac{5t}{t + 2}\right)$
28. $\lim\limits_{x\to\infty}\left[\dfrac{x}{2x + 1} + \dfrac{3x^2}{(x - 3)^2}\right]$

In Exercises 29–34, use a graphing utility to graph the function and verify that the horizontal asymptote corresponds to the limit at infinity.

29. $y = \dfrac{3x}{1 - x}$
30. $y = \dfrac{x^2}{x^2 + 4}$
31. $y = \dfrac{2x}{1 - x^2}$
32. $y = \dfrac{2x + 1}{x^2 - 1}$
33. $y = 1 - \dfrac{3}{x^2}$
34. $y = 2 + \dfrac{1}{x}$

NUMERICAL AND GRAPHICAL ANALYSIS In Exercises 35–38, (a) complete the table and numerically estimate the limit as x approaches infinity, and (b) use a graphing utility to graph the function and estimate the limit graphically.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

35. $f(x) = x - \sqrt{x^2 + 2}$
36. $f(x) = 3x - \sqrt{9x^2 + 1}$
37. $f(x) = 3\left(2x - \sqrt{4x^2 + x}\right)$
38. $f(x) = 4\left(4x - \sqrt{16x^2 - x}\right)$

In Exercises 39–48, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume n begins with 1.

39. $a_n = \dfrac{n+1}{n^2+1}$

40. $a_n = \dfrac{3n}{n^2+2}$

41. $a_n = \dfrac{n}{2n+1}$

42. $a_n = \dfrac{4n-1}{n+3}$

43. $a_n = \dfrac{n^2}{2n+3}$

44. $a_n = \dfrac{4n^2+1}{2n}$

45. $a_n = \dfrac{(n+1)!}{n!}$

46. $a_n = \dfrac{(3n-1)!}{(3n+1)!}$

47. $a_n = \dfrac{(-1)^n}{n}$

48. $a_n = \dfrac{(-1)^{n+1}}{n^2}$

In Exercises 49–52, find the limit of the sequence. Then verify the limit numerically by using a graphing utility to complete the table.

n	10^0	10^1	10^2	10^3	10^4	10^5	10^6
a_n							

49. $a_n = \dfrac{1}{n}\left(n + \dfrac{1}{n}\left[\dfrac{n(n+1)}{2}\right]\right)$

50. $a_n = \dfrac{4}{n}\left(n + \dfrac{4}{n}\left[\dfrac{n(n+1)}{2}\right]\right)$

51. $a_n = \dfrac{16}{n^3}\left[\dfrac{n(n+1)(2n+1)}{6}\right]$

52. $a_n = \dfrac{n(n+1)}{n^2} - \dfrac{1}{n^4}\left[\dfrac{n(n+1)}{2}\right]^2$

53. OXYGEN LEVEL Suppose that $f(t)$ measures the level of oxygen in a pond, where $f(t) = 1$ is the normal (unpolluted) level and the time t is measured in weeks. When $t = 0$, organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is given by

$$f(t) = \frac{t^2 - t + 1}{t^2 + 1}.$$

(a) What is the limit of f as t approaches infinity?

(b) Use a graphing utility to graph the function and verify the result of part (a).

(c) Explain the meaning of the limit in the context of the problem.

54. TYPING SPEED The average typing speed S (in words per minute) for a student after t weeks of lessons is given by

$$S = \frac{100t^2}{65 + t^2}, \quad t > 0.$$

(a) What is the limit of S as t approaches infinity?

(b) Use a graphing utility to graph the function and verify the result of part (a).

(c) Explain the meaning of the limit in the context of the problem.

55. AVERAGE COST The cost function for a certain model of personal digital assistant (PDA) is given by $C = 13.50x + 45{,}750$, where C is measured in dollars and x is the number of PDAs produced.

(a) Write a model for the average cost per unit produced.

(b) Find the average costs per unit when $x = 100$ and $x = 1000$.

(c) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

56. AVERAGE COST The cost function for a company to recycle x tons of material is given by $C = 1.25x + 10{,}500$, where C is measured in dollars.

(a) Write a model for the average cost per ton of material recycled.

(b) Find the average costs of recycling 100 tons of material and 1000 tons of material.

(c) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

57. DATA ANALYSIS: SOCIAL SECURITY The table shows the average monthly Social Security benefits B (in dollars) for retired workers aged 62 or over from 2001 through 2007. (Source: U.S. Social Security Administration)

Year	Benefit, B
2001	874
2002	895
2003	922
2004	955
2005	1002
2006	1044
2007	1079

A model for the data is given by

$$B = \frac{867.3 + 707.56t}{1.0 + 0.83t - 0.030t^2}, \quad 1 \le t \le 7$$

where t represents the year, with $t = 1$ corresponding to 2001.

(a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How well does the model fit the data?

(b) Use the model to predict the average monthly benefit in 2014.

(c) Discuss why this model should not be used for long-term predictions of average monthly Social Security benefits.

58. DATA ANALYSIS: MILITARY The table shows the numbers N (in thousands) of U.S. military reserve personnel for the years 2001 through 2007. (Source: U.S. Department of Defense)

Year	Number, N
2001	1249
2002	1222
2003	1189
2004	1167
2005	1136
2006	1120
2007	1110

A model for the data is given by

$$N = \frac{1287.9 + 61.53t}{1.0 + 0.08t}, \quad 1 \le t \le 7$$

where t represents the year, with $t = 1$ corresponding to 2001.

(a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How well does the model fit the data?

(b) Use the model to predict the number of military reserve personnel in 2014.

(c) What is the limit of the function as t approaches infinity? Explain the meaning of the limit in the context of the problem. Do you think the limit is realistic? Explain.

EXPLORATION

TRUE OR FALSE? **In Exercises 59–62, determine whether the statement is true or false. Justify your answer.**

59. Every rational function has a horizontal asymptote.

60. If $f(x)$ increases without bound as x approaches c, then the limit of $f(x)$ exists.

61. If a sequence converges, then it has a limit.

62. When the degrees of the numerator and denominator of a rational function are equal, the limit does not exist.

63. THINK ABOUT IT Find the functions f and g such that both $f(x)$ and $g(x)$ increase without bound as x approaches c, but $\lim_{x \to c}[f(x) - g(x)]$ exists.

64. THINK ABOUT IT Use a graphing utility to graph the function given by

$$f(x) = \frac{x}{\sqrt{x^2 + 1}}.$$

How many horizontal asymptotes does the function appear to have? What are the horizontal asymptotes?

In Exercises 65–68, create a scatter plot of the terms of the sequence. Determine whether the sequence converges or diverges. If it converges, estimate its limit.

65. $a_n = 4\left(\frac{2}{3}\right)^n$

66. $a_n = 3\left(\frac{3}{2}\right)^n$

67. $a_n = \dfrac{3[1 - (1.5)^n]}{1 - 1.5}$

68. $a_n = \dfrac{3[1 - (0.5)^n]}{1 - 0.5}$

69. Use a graphing utility to graph the two functions given by

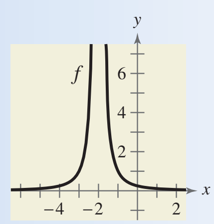
$$y_1 = \frac{1}{\sqrt{x}} \quad \text{and} \quad y_2 = \frac{1}{\sqrt[3]{x}}$$

in the same viewing window. Why does y_1 not appear to the left of the y-axis? How does this relate to the statement at the top of page 882 about the infinite limit

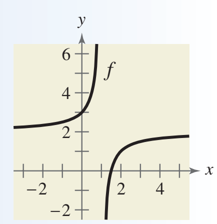
$$\lim_{x \to -\infty} \frac{1}{x^r}?$$

70. CAPSTONE Use the graph to estimate (a) $\lim_{x \to \infty} f(x)$, (b) $\lim_{x \to -\infty} f(x)$, and (c) the horizontal asymptote of the graph of f.

(i)



(ii)



71. Use a graphing utility to complete the table below to verify that $\lim_{x \to \infty} (1/x) = 0$.

x	10^0	10^1	10^2	10^3	10^4	10^5
$\frac{1}{x}$						

Make a conjecture about $\lim_{x \to 0} \dfrac{1}{x}$.