

12.4 LIMITS AT INFINITY AND LIMITS OF SEQUENCES

What you should learn

- Evaluate limits of functions at infinity.
- Find limits of sequences.

Why you should learn it

Finding limits at infinity is useful in many types of real-life applications. For instance, in Exercise 58 on page 889, you are asked to find a limit at infinity to determine the number of military reserve personnel in the future.



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Algebra Help

The function $f(x) = \frac{x+1}{2x}$ is a rational function. You can review rational functions in Section 2.6.

Limits at Infinity and Horizontal Asymptotes

As pointed out at the beginning of this chapter, there are two basic problems in calculus: finding **tangent lines** and finding the area of a region. In Section 12.3, you saw how limits can be used to solve the tangent line problem. In this section and the next, you will see how a different type of limit, a *limit at infinity*, can be used to solve the area problem. To get an idea of what is meant by a limit at infinity, consider the function given by

$$f(x) = \frac{x+1}{2x}.$$

The graph of f is shown in Figure 12.30. From earlier work, you know that $y = \frac{1}{2}$ is a horizontal asymptote of the graph of this function. Using limit notation, this can be written as follows.

$$\lim_{x \rightarrow -\infty} f(x) = \frac{1}{2} \quad \text{Horizontal asymptote to the left}$$

$$\lim_{x \rightarrow \infty} f(x) = \frac{1}{2} \quad \text{Horizontal asymptote to the right}$$

These limits mean that the value of $f(x)$ gets arbitrarily close to $\frac{1}{2}$ as x decreases or increases without bound.

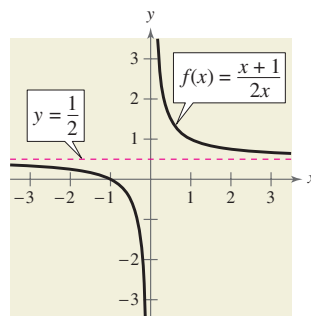


FIGURE 12.30

Definition of Limits at Infinity

If f is a function and L_1 and L_2 are real numbers, the statements

$$\lim_{x \rightarrow -\infty} f(x) = L_1 \quad \text{Limit as } x \text{ approaches } -\infty$$

and

$$\lim_{x \rightarrow \infty} f(x) = L_2 \quad \text{Limit as } x \text{ approaches } \infty$$

denote the **limits at infinity**. The first statement is read “the limit of $f(x)$ as x approaches $-\infty$ is L_1 ,” and the second is read “the limit of $f(x)$ as x approaches ∞ is L_2 .”

To help evaluate limits at infinity, you can use the following definition.

Limits at Infinity

If r is a positive real number, then

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0. \quad \text{Limit toward the right}$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0. \quad \text{Limit toward the left}$$

Limits at infinity share many of the properties of limits listed in Section 12.1. Some of these properties are demonstrated in the next example.

Example 1 Evaluating a Limit at Infinity

Find the limit.

$$\lim_{x \rightarrow \infty} \left(4 - \frac{3}{x^2} \right)$$

Algebraic Solution

Use the properties of limits listed in Section 12.1.

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(4 - \frac{3}{x^2} \right) &= \lim_{x \rightarrow \infty} 4 - \lim_{x \rightarrow \infty} \frac{3}{x^2} \\ &= \lim_{x \rightarrow \infty} 4 - 3 \left(\lim_{x \rightarrow \infty} \frac{1}{x^2} \right) \\ &= 4 - 3(0) \\ &= 4 \end{aligned}$$

So, the limit of $f(x) = 4 - \frac{3}{x^2}$ as x approaches ∞ is 4.

CheckPoint Now try Exercise 9.

Graphical Solution

Use a graphing utility to graph $y = 4 - 3/x^2$. Then use the *trace* feature to determine that as x gets larger and larger, y gets closer and closer to 4, as shown in Figure 12.31. Note that the line $y = 4$ is a horizontal asymptote to the right.

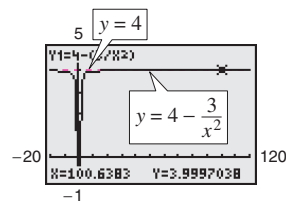


FIGURE 12.31

In Figure 12.31, it appears that the line $y = 4$ is also a horizontal asymptote to the left. You can verify this by showing that

$$\lim_{x \rightarrow -\infty} \left(4 - \frac{3}{x^2} \right) = 4.$$

The graph of a rational function need not have a horizontal asymptote. If it does, however, its left and right horizontal asymptotes must be the same.

When evaluating limits at infinity for more complicated rational functions, divide the numerator and denominator by the *highest-powered term* in the denominator. This enables you to evaluate each limit using the limits at infinity at the top of this page.

Example 2 Comparing Limits at Infinity

Find the limit as x approaches ∞ for each function.

$$\text{a. } f(x) = \frac{-2x + 3}{3x^2 + 1} \quad \text{b. } f(x) = \frac{-2x^2 + 3}{3x^2 + 1} \quad \text{c. } f(x) = \frac{-2x^3 + 3}{3x^2 + 1}$$

Solution

In each case, begin by dividing both the numerator and denominator by x^2 , the highest-powered term in the denominator.

$$\begin{aligned} \text{a. } \lim_{x \rightarrow \infty} \frac{-2x + 3}{3x^2 + 1} &= \lim_{x \rightarrow \infty} \frac{-\frac{2}{x} + \frac{3}{x^2}}{3 + \frac{1}{x^2}} \\ &= \frac{-0 + 0}{3 + 0} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{b. } \lim_{x \rightarrow \infty} \frac{-2x^2 + 3}{3x^2 + 1} &= \lim_{x \rightarrow \infty} \frac{-2 + \frac{3}{x^2}}{3 + \frac{1}{x^2}} \\ &= \frac{-2 + 0}{3 + 0} \\ &= -\frac{2}{3} \end{aligned}$$

$$\text{c. } \lim_{x \rightarrow \infty} \frac{-2x^3 + 3}{3x^2 + 1} = \lim_{x \rightarrow \infty} \frac{-2x + \frac{3}{x^2}}{3 + \frac{1}{x^2}}$$

In this case, you can conclude that the limit does not exist because the numerator decreases without bound as the denominator approaches 3.

CHECKPOINT Now try Exercise 19.

In Example 2, observe that when the degree of the numerator is less than the degree of the denominator, as in part (a), the limit is 0. When the degrees of the numerator and denominator are equal, as in part (b), the limit is the ratio of the coefficients of the highest-powered terms. When the degree of the numerator is greater than the degree of the denominator, as in part (c), the limit does not exist.

This result seems reasonable when you realize that for large values of x , the highest-powered term of a polynomial is the most “influential” term. That is, a polynomial tends to behave as its highest-powered term behaves as x approaches positive or negative infinity.

Limits at Infinity for Rational Functions

Consider the rational function $f(x) = N(x)/D(x)$, where

$$N(x) = a_n x^n + \cdots + a_0 \quad \text{and} \quad D(x) = b_m x^m + \cdots + b_0.$$

The limit of $f(x)$ as x approaches positive or negative infinity is as follows.

$$\lim_{x \rightarrow \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, the limit does not exist.

Example 3 Finding the Average Cost

You are manufacturing greeting cards that cost \$0.50 per card to produce. Your initial investment is \$5000, which implies that the total cost C of producing x cards is given by $C = 0.50x + 5000$. The average cost \bar{C} per card is given by

$$\bar{C} = \frac{C}{x} = \frac{0.50x + 5000}{x}.$$

Find the average cost per card when (a) $x = 1000$, (b) $x = 10,000$, and (c) $x = 100,000$. (d) What is the limit of \bar{C} as x approaches infinity?

Solution

a. When $x = 1000$, the average cost per card is

$$\begin{aligned} \bar{C} &= \frac{0.50(1000) + 5000}{1000} && x = 1000 \\ &= \$5.50. \end{aligned}$$

b. When $x = 10,000$, the average cost per card is

$$\begin{aligned} \bar{C} &= \frac{0.50(10,000) + 5000}{10,000} && x = 10,000 \\ &= \$1.00. \end{aligned}$$

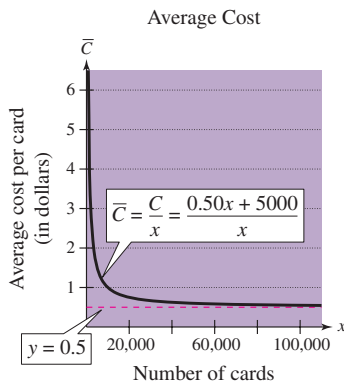
c. When $x = 100,000$, the average cost per card is

$$\begin{aligned} \bar{C} &= \frac{0.50(100,000) + 5000}{100,000} && x = 100,000 \\ &= \$0.55. \end{aligned}$$

d. As x approaches infinity, the limit of \bar{C} is

$$\lim_{x \rightarrow \infty} \frac{0.50x + 5000}{x} = \$0.50. \quad x \rightarrow \infty$$

The graph of \bar{C} is shown in Figure 12.32.



As $x \rightarrow \infty$, the average cost per card approaches \$0.50.

FIGURE 12.32

CHECKPOINT Now try Exercise 55.

Algebra Help

You can review sequences in Sections 9.1–9.3.

TECHNOLOGY

There are a number of ways to use a graphing utility to generate the terms of a sequence. For instance, you can display the first 10 terms of the sequence

$$a_n = \frac{1}{2^n}$$

using the *sequence* feature or the *table* feature.

Limits of Sequences

Limits of sequences have many of the same properties as limits of functions. For instance, consider the sequence whose n th term is $a_n = 1/2^n$.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$$

As n increases without bound, the terms of this sequence get closer and closer to 0, and the sequence is said to **converge** to 0. Using limit notation, you can write

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0.$$

The following relationship shows how limits of functions of x can be used to evaluate the limit of a sequence.

Limit of a Sequence

Let f be a function of a real variable such that

$$\lim_{x \rightarrow \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n , then

$$\lim_{n \rightarrow \infty} a_n = L.$$

A sequence that does not converge is said to **diverge**. For instance, the terms of the sequence $1, -1, 1, -1, 1, \dots$ oscillate between 1 and -1 . Therefore, the sequence diverges because it does not approach a unique number.

Example 4 Finding the Limit of a Sequence

Find the limit of each sequence. (Assume n begins with 1.)

a. $a_n = \frac{2n + 1}{n + 4}$

b. $b_n = \frac{2n + 1}{n^2 + 4}$

c. $c_n = \frac{2n^2 + 1}{4n^2}$

Solution

a. $\lim_{n \rightarrow \infty} \frac{2n + 1}{n + 4} = 2$ $\frac{3}{5}, \frac{5}{6}, \frac{7}{7}, \frac{9}{8}, \frac{11}{9}, \frac{13}{10}, \dots \rightarrow 2$

b. $\lim_{n \rightarrow \infty} \frac{2n + 1}{n^2 + 4} = 0$ $\frac{3}{5}, \frac{5}{8}, \frac{7}{13}, \frac{9}{20}, \frac{11}{29}, \frac{13}{40}, \dots \rightarrow 0$

c. $\lim_{n \rightarrow \infty} \frac{2n^2 + 1}{4n^2} = \frac{1}{2}$ $\frac{3}{4}, \frac{9}{16}, \frac{19}{36}, \frac{33}{64}, \frac{51}{100}, \frac{73}{144}, \dots \rightarrow \frac{1}{2}$

CHECKPOINT Now try Exercise 39.

Study Tip

You can use the definition of limits at infinity for rational functions on page 884 to verify the limits of the sequences in Example 4.

In the next section, you will encounter limits of sequences such as that shown in Example 5. A strategy for evaluating such limits is to begin by writing the n th term in standard rational function form. Then you can determine the limit by comparing the degrees of the numerator and denominator, as shown on page 884.

Example 5 Finding the Limit of a Sequence

Find the limit of the sequence whose n th term is

$$a_n = \frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right].$$

Algebraic Solution

Begin by writing the n th term in standard rational function form—as the ratio of two polynomials.

$$\begin{aligned} a_n &= \frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] && \text{Write original } n\text{th term.} \\ &= \frac{8(n)(n+1)(2n+1)}{6n^3} && \text{Multiply fractions.} \\ &= \frac{8n^3 + 12n^2 + 4n}{3n^3} && \text{Write in standard rational form.} \end{aligned}$$

From this form, you can see that the degree of the numerator is equal to the degree of the denominator. So, the limit of the sequence is the ratio of the coefficients of the highest-powered terms.

$$\lim_{n \rightarrow \infty} \frac{8n^3 + 12n^2 + 4n}{3n^3} = \frac{8}{3}$$

CHECKPOINT Now try Exercise 49.

Numerical Solution

Construct a table that shows the value of a_n as n becomes larger and larger, as shown below.

n	a_n
1	8
10	3.08
100	2.707
1000	2.671
10,000	2.667

From the table, you can estimate that as n approaches ∞ , a_n gets closer and closer to $2.667 \approx \frac{8}{3}$.

CLASSROOM DISCUSSION

Comparing Rates of Convergence In the table in Example 5 above, the value of a_n approaches its limit of $\frac{8}{3}$ rather slowly. (The first term to be accurate to three decimal places is $a_{4801} \approx 2.667$.) Each of the following sequences converges to 0. Which converges the quickest? Which converges the slowest? Why? Write a short paragraph discussing your conclusions.

- a. $a_n = \frac{1}{n}$ b. $b_n = \frac{1}{n^2}$ c. $c_n = \frac{1}{2^n}$
 d. $d_n = \frac{1}{n!}$ e. $h_n = \frac{2^n}{n!}$