12.2 TECHNIQUES FOR EVALUATING LIMITS

What you should learn

- Use the dividing out technique to evaluate limits of functions.
- Use the rationalizing technique to evaluate limits of functions.
- Approximate limits of functions graphically and numerically.
- Evaluate one-sided limits of functions.
- Evaluate limits of difference quotients from calculus.

Why you should learn it

Limits can be applied in real-life situations. For instance, in Exercise 84 on page 870, you will determine limits involving the costs of making photocopies.

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Dividing Out Technique

In Section 12.1, you studied several types of functions whose limits can be evaluated by direct substitution. In this section, you will study several techniques for evaluating limits of functions for which direct substitution fails.

Suppose you were asked to find the following limit.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Direct substitution produces 0 in both the numerator and denominator.

$$(-3)^2 + (-3) - 6 = 0 \qquad \text{Numerator is 0 when } x = -3.$$

$$-3 + 3 = 0 \qquad \text{Denominator is 0 when } x = -3.$$

The resulting fraction, $\frac{0}{0}$, has no meaning as a real number. It is called an **indeterminate form** because you cannot, from the form alone, determine the limit. By using a table, however, it appears that the limit of the function as $x \to -3$ is -5.

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\frac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	?	-4.9999	-4.999	-4.99

When you try to evaluate a limit of a rational function by direct substitution and encounter the indeterminate form $\frac{0}{0}$, you can conclude that the numerator and denominator must have a common factor. After factoring and dividing out, you should try direct substitution again. Example 1 shows how you can use the **dividing out technique** to evaluate limits of these types of functions.

Example 1 Dividing Out Technique

Find the limit: $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$.

Solution

From the discussion above, you know that direct substitution fails. So, begin by factoring the numerator and dividing out any common factors.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \qquad \text{Factor numerator.}$$

$$= \lim_{x \to -3} \frac{(x - 2)\cancel{(x + 3)}}{\cancel{x + 3}} \qquad \text{Divide out common factor.}$$

$$= \lim_{x \to -3} (x - 2) \qquad \text{Simplify.}$$

$$= -3 - 2 = -5 \qquad \text{Direct substitution and simplify.}$$

CHECKPoint Now try Exercise 11.

The validity of the dividing out technique stems from the fact that if two functions agree at all but a single number c, they must have identical limit behavior at $x = c$. In Example 1, the functions given by

$$f(x) = \frac{x^2 + x - 6}{x + 3} \quad \text{and} \quad g(x) = x - 2$$

agree at all values of x other than $x = -3$. So, you can use $g(x)$ to find the limit of $f(x)$.

Example 2 Dividing Out Technique

Find the limit.

$$\lim_{x\to 1} \frac{x - 1}{x^3 - x^2 + x - 1}$$

Solution

Begin by substituting $x = 1$ into the numerator and denominator.

$$1 - 1 = 0 \qquad \text{Numerator is 0 when } x = 1.$$

$$1^3 - 1^2 + 1 - 1 = 0 \qquad \text{Denominator is 0 when } x = 1.$$

Because both the numerator and denominator are zero when $x = 1$, direct substitution will not yield the limit. To find the limit, you should factor the numerator and denominator, divide out any common factors, and then try direct substitution again.

$$\lim_{x\to 1} \frac{x - 1}{x^3 - x^2 + x - 1} = \lim_{x\to 1} \frac{x - 1}{(x - 1)(x^2 + 1)} \qquad \text{Factor denominator.}$$

$$= \lim_{x\to 1} \frac{\cancel{x - 1}}{\cancel{(x - 1)}(x^2 + 1)} \qquad \text{Divide out common factor.}$$

$$= \lim_{x\to 1} \frac{1}{x^2 + 1} \qquad \text{Simplify.}$$

$$= \frac{1}{1^2 + 1} \qquad \text{Direct substitution}$$

$$= \frac{1}{2} \qquad \text{Simplify.}$$

This result is shown graphically in Figure 12.11.

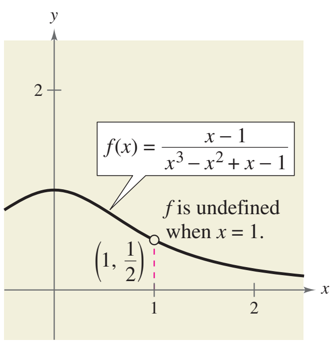

FIGURE 12.11

CHECKPoint Now try Exercise 15.

In Example 2, the factorization of the denominator can be obtained by dividing by $(x - 1)$ or by grouping as follows.

$$x^3 - x^2 + x - 1 = x^2(x - 1) + (x - 1)$$

$$= (x - 1)(x^2 + 1)$$

Rationalizing Technique

Algebra Help

You can review the techniques for rationalizing numerators and denominators in Appendix A.2.

Another way to find the limits of some functions is first to rationalize the numerator of the function. This is called the **rationalizing technique.** Recall that rationalizing the numerator means multiplying the numerator and denominator by the conjugate of the numerator. For instance, the conjugate of $\sqrt{x} + 4$ is $\sqrt{x} - 4$.

Example 3 Rationalizing Technique

Find the limit: $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Solution

By direct substitution, you obtain the indeterminate form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{0+1}-1}{0} = \frac{0}{0} \qquad \text{Indeterminate form}$$

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}
\frac{\sqrt{x+1}-1}{x} &= \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \\
&= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)} && \text{Multiply.} \\
&= \frac{x}{x\left(\sqrt{x+1}+1\right)} && \text{Simplify.} \\
&= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)} && \text{Divide out common factor.} \\
&= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0 && \text{Simplify.}
\end{aligned}$$

Now you can evaluate the limit by direct substitution.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{1+1} = \frac{1}{2}$$

You can reinforce your conclusion that the limit is $\frac{1}{2}$ by constructing a table, as shown below, or by sketching a graph, as shown in Figure 12.12.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

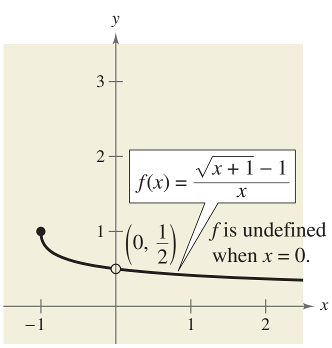

FIGURE 12.12

CHECKPoint Now try Exercise 25.

The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 3, the convenient form is

$$1 = \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}.$$

Using Technology

The dividing out and rationalizing techniques may not work well for finding limits of nonalgebraic functions. You often need to use more sophisticated analytic techniques to find limits of these types of functions.

Example 4 Approximating a Limit

Approximate the limit: $\lim_{x\to 0} (1 + x)^{1/x}$.

Numerical Solution

Let $f(x) = (1 + x)^{1/x}$. Because you are finding the limit when $x = 0$, use the *table* feature of a graphing utility to create a table that shows the values of f for x starting at $x = -0.01$ and has a step of 0.001, as shown in Figure 12.13. Because 0 is halfway between -0.001 and 0.001, use the average of the values of f at these two x-coordinates to estimate the limit, as follows.

$$\lim_{x\to 0} (1 + x)^{1/x} \approx \frac{2.7196 + 2.7169}{2} = 2.71825$$

The actual limit can be found algebraically to be $e \approx 2.71828$.

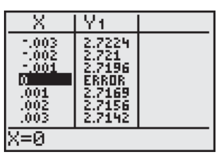

FIGURE 12.13

CHECK*Point* Now try Exercise 37.

Graphical Solution

To approximate the limit graphically, graph the function $f(x) = (1 + x)^{1/x}$, as shown in Figure 12.14. Using the *zoom* and *trace* features of the graphing utility, choose two points on the graph of f, such as

$$(-0.00017, 2.7185) \quad \text{and} \quad (0.00017, 2.7181)$$

as shown in Figure 12.15. Because the x-coordinates of these two points are equidistant from 0, you can approximate the limit to be the average of the y-coordinates. That is,

$$\lim_{x\to 0} (1 + x)^{1/x} \approx \frac{2.7185 + 2.7181}{2} = 2.7183.$$

The actual limit can be found algebraically to be $e \approx 2.71828$.

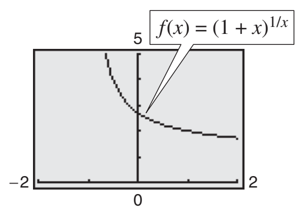

FIGURE 12.14

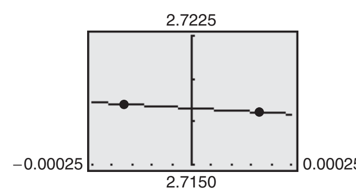

FIGURE 12.15

Example 5 Approximating a Limit Graphically

Approximate the limit: $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution

Direct substitution produces the indeterminate form $\frac{0}{0}$. To approximate the limit, begin by using a graphing utility to graph $f(x) = (\sin x)/x$, as shown in Figure 12.16. Then use the *zoom* and *trace* features of the graphing utility to choose a point on each side of 0, such as $(-0.0012467, 0.9999997)$ and $(0.0012467, 0.9999997)$. Finally, approximate the limit as the average of the y-coordinates of these two points, $\lim_{x\to 0} (\sin x)/x \approx 0.9999997$. It can be shown algebraically that this limit is exactly 1.

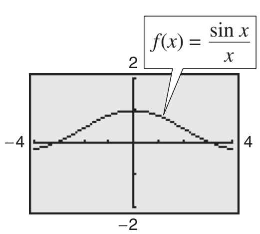

FIGURE 12.16

CHECK*Point* Now try Exercise 41.

TECHNOLOGY

The graphs shown in Figures 12.14 and 12.16 appear to be continuous at $x = 0$. However, when you try to use the *trace* or the *value* feature of a graphing utility to determine the value of y when $x = 0$, no value is given. Some graphing utilities can show breaks or holes in a graph when an appropriate viewing window is used. Because the holes in the graphs in Figures 12.14 and 12.16 occur on the y-axis, the holes are not visible.

One-Sided Limits

In Section 12.1, you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left side of c than it approaches from the right side of c. This type of behavior can be described more concisely with the concept of a **one-sided limit.**

$$\lim_{x \to c^-} f(x) = L_1 \quad \text{or} \quad f(x) \to L_1 \text{ as } x \to c^- \qquad \text{Limit from the left}$$

$$\lim_{x \to c^+} f(x) = L_2 \quad \text{or} \quad f(x) \to L_2 \text{ as } x \to c^+ \qquad \text{Limit from the right}$$

Example 6 Evaluating One-Sided Limits

Find the limit as $x \to 0$ from the left and the limit as $x \to 0$ from the right for

$$f(x) = \frac{|2x|}{x}.$$

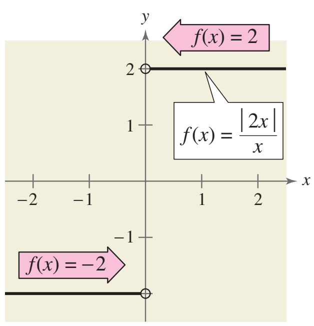

FIGURE 12.17

Solution

From the graph of f, shown in Figure 12.17, you can see that $f(x) = -2$ for all $x < 0$. Therefore, the limit from the left is

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2. \qquad \text{Limit from the left: } f(x) \to -2 \text{ as } x \to 0^-$$

Because $f(x) = 2$ for all $x > 0$, the limit from the right is

$$\lim_{x \to 0^+} \frac{|2x|}{x} = 2. \qquad \text{Limit from the right: } f(x) \to 2 \text{ as } x \to 0^+$$

CHECK*Point* Now try Exercise 55.

In Example 6, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x \to c$ does not exist. For the limit of a function to exist as $x \to c$, it must be true that both one-sided limits exist and are equal.

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \to c} f(x) = L$$

if and only if both the left and right limits *exist* and are *equal* to L.

Example 7 Finding One-Sided Limits

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

Solution

Remember that you are concerned about the value of f *near* $x = 1$ rather than *at* $x = 1$. So, for $x < 1$, $f(x)$ is given by $4 - x$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (4 - x) \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

For $x > 1$, $f(x)$ is given by $4x - x^2$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (4x - x^2) \\ &= 4(1) - 1^2 \\ &= 3. \end{aligned}$$

Because the one-sided limits both exist and are equal to 3, it follows that

$$\lim_{x\to 1} f(x) = 3.$$

The graph in Figure 12.18 confirms this conclusion.

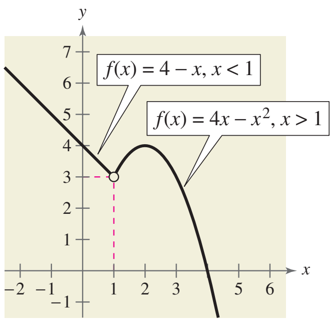

FIGURE 12.18

CHECKPoint Now try Exercise 59.

Example 8 Comparing Limits from the Left and Right

To ship a package overnight, a delivery service charges \$18 for the first pound and \$2 for each additional pound or portion of a pound. Let x represent the weight of a package and let $f(x)$ represent the shipping cost. Show that the limit of $f(x)$ as $x \to 2$ does not exist.

$$f(x) = \begin{cases} \$18, & 0 < x \le 1 \\ \$20, & 1 < x \le 2 \\ \$22, & 2 < x \le 3 \end{cases}$$

Solution

The graph of f is shown in Figure 12.19. The limit of $f(x)$ as x approaches 2 from the left is

$$\lim_{x\to 2^-} f(x) = 20$$

whereas the limit of $f(x)$ as x approaches 2 from the right is

$$\lim_{x\to 2^+} f(x) = 22.$$

Because these one-sided limits are not equal, the limit of $f(x)$ as $x \to 2$ does not exist.

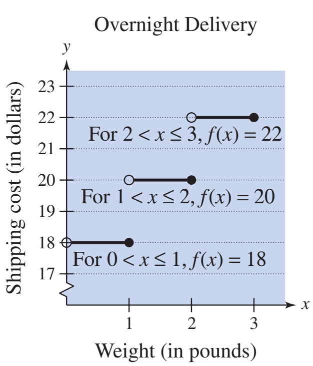

FIGURE 12.19

CHECKPoint Now try Exercise 81.

A Limit from Calculus

In the next section, you will study an important type of limit from calculus—the limit of a **difference quotient.**

Example 9 Evaluating a Limit from Calculus

For the function given by $f(x) = x^2 - 1$, find

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}.$$

Solution

Direct substitution produces an indeterminate form.

$$\begin{aligned}\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \to 0} \frac{[(3+h)^2 - 1] - [(3)^2 - 1]}{h} \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h} \\ &= \lim_{h \to 0} \frac{6h + h^2}{h} \\ &= \frac{0}{0}\end{aligned}$$

By factoring and dividing out, you obtain the following.

$$\begin{aligned}\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{6h + h^2}{h} &= \lim_{h \to 0} \frac{\cancel{h}(6+h)}{\cancel{h}} \\ &= \lim_{h \to 0} (6 + h) \\ &= 6 + 0 \\ &= 6\end{aligned}$$

So, the limit is 6.

CHECK*Point* Now try Exercise 75.

Note that for any x-value, the limit of a difference quotient is an expression of the form

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Direct substitution into the difference quotient always produces the indeterminate form $\frac{0}{0}$. For instance,

$$\begin{aligned}\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \frac{f(x+0) - f(x)}{0} \\ &= \frac{f(x) - f(x)}{0} \\ &= \frac{0}{0}.\end{aligned}$$

Algebra Help

For a review of evaluating difference quotients, refer to Section 1.4.