12.1 Introduction to Limits

What you should learn

- Use the definition of limit to estimate limits.
- Determine whether limits of functions exist.
- Use properties of limits and direct substitution to evaluate limits.

Why you should learn it

The concept of a limit is useful in applications involving maximization. For instance, in Exercise 5 on page 858, the concept of a limit is used to verify the maximum volume of an open box.

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The Limit Concept

The notion of a limit is a *fundamental* concept of calculus. In this chapter, you will learn how to evaluate limits and how they are used in the two basic problems of calculus: the tangent line problem and the area problem.

Example 1 Finding a Rectangle of Maximum Area

You are given 24 inches of wire and are asked to form a rectangle whose area is as large as possible. Determine the dimensions of the rectangle that will produce a maximum area.

Solution

Let w represent the width of the rectangle and let l represent the length of the rectangle. Because

$$2w + 2l = 24 \qquad \text{Perimeter is 24.}$$

it follows that $l = 12 - w$, as shown in Figure 12.1. So, the area of the rectangle is

$$A = lw \qquad \text{Formula for area}$$
$$= (12 - w)w \qquad \text{Substitute } 12 - w \text{ for } l.$$
$$= 12w - w^2. \qquad \text{Simplify.}$$

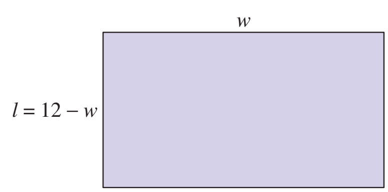

FIGURE 12.1

Using this model for area, you can experiment with different values of w to see how to obtain the maximum area. After trying several values, it appears that the maximum area occurs when $w = 6$, as shown in the table.

Width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	35.75	35.99	36.00	35.99	35.75	35.00

In limit terminology, you can say that "the limit of A as w approaches 6 is 36." This is written as

$$\lim_{w \to 6} A = \lim_{w \to 6} (12w - w^2) = 36.$$

CHECKPoint Now try Exercise 5.

Study Tip

An alternative notation for $\lim_{x\to c} f(x) = L$ is

$$f(x) \to L \text{ as } x \to c$$

which is read as "$f(x)$ approaches L as x approaches c."

Definition of Limit

Definition of Limit

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, the limit of $f(x)$ as x approaches c is L. This is written as

$$\lim_{x\to c} f(x) = L.$$

Example 2 Estimating a Limit Numerically

Use a table to estimate numerically the limit: $\lim_{x\to 2} (3x - 2)$.

Solution

Let $f(x) = 3x - 2$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one set that approaches 2 from the left and one that approaches 2 from the right.

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300

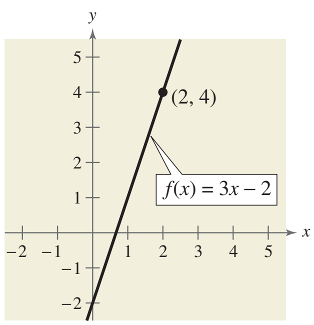

FIGURE 12.2

From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4. So, you can estimate the limit to be 4. Figure 12.2 adds further support for this conclusion.

CHECKPoint Now try Exercise 7.

In Figure 12.2, note that the graph of $f(x) = 3x - 2$ is continuous. For graphs that are not continuous, finding a limit can be more difficult.

Example 3 Estimating a Limit Numerically

Use a table to estimate numerically the limit: $\lim_{x\to 0} \dfrac{x}{\sqrt{x+1} - 1}$.

Solution

Let $f(x) = x/\left(\sqrt{x+1} - 1\right)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one set that approaches 0 from the left and one that approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99949	1.99995	?	2.00005	2.00050	2.00499

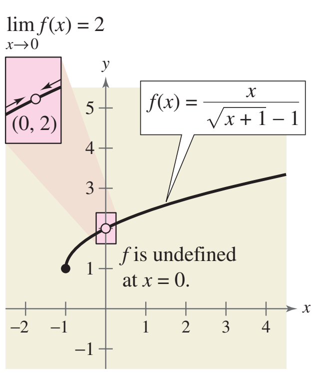

FIGURE 12.3

From the table, it appears that the limit is 2. The graph shown in Figure 12.3 verifies that the limit is 2.

CHECKPoint Now try Exercise 9.

In Example 3, note that $f(x)$ has a limit when $x \to 0$ even though the function is not defined when $x = 0$. This often happens, and it is important to realize that *the existence or nonexistence of* $f(x)$ *at* $x = c$ *has no bearing on the existence of the limit of* $f(x)$ *as* x *approaches* c.

Example 4 Estimating a Limit

Estimate the limit: $\lim_{x\to 1} \dfrac{x^3 - x^2 + x - 1}{x - 1}$.

Numerical Solution

Let $f(x) = (x^3 - x^2 + x - 1)/(x - 1)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one set that approaches 1 from the left and one that approaches 1 from the right.

x	0.9	0.99	0.999	1.0
$f(x)$	1.8100	1.9801	1.9980	?

x	1.0	1.001	1.01	1.1
$f(x)$	?	2.0020	2.0201	2.2100

From the tables, it appears that the limit is 2.

Graphical Solution

Let $f(x) = (x^3 - x^2 + x - 1)/(x - 1)$. Then sketch a graph of the function, as shown in Figure 12.4. From the graph, it appears that as x approaches 1 from either side, $f(x)$ approaches 2. So, you can estimate the limit to be 2.

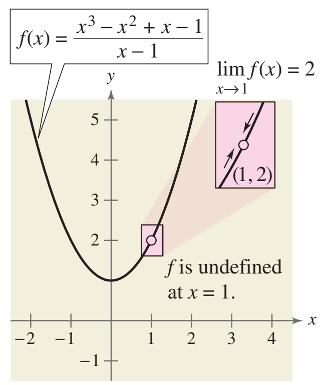

FIGURE 12.4

CHECKPoint Now try Exercise 13.

Example 5 Using a Graph to Find a Limit

Find the limit of $f(x)$ as x approaches 3, where f is defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 0, & x = 3 \end{cases}.$$

Solution

Because $f(x) = 2$ for all x other than $x = 3$ and because the value of $f(3)$ is immaterial, it follows that the limit is 2 (see Figure 12.5). So, you can write

$$\lim_{x\to 3} f(x) = 2.$$

The fact that $f(3) = 0$ has no bearing on the existence or value of the limit as x approaches 3. For instance, if the function were defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

the limit as x approaches 3 would be the same.

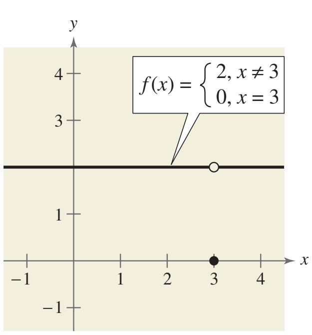

FIGURE 12.5

CHECKPoint Now try Exercise 27.

Limits That Fail to Exist

Next, you will examine some functions for which limits do not exist.

Example 6 Comparing Left and Right Behavior

Show that the limit does not exist.

$$\lim_{x \to 0} \frac{|x|}{x}$$

Solution

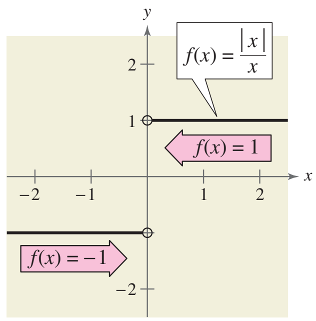

FIGURE 12.6

Consider the graph of the function given by $f(x) = |x|/x$. From Figure 12.6, you can see that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \quad x < 0.$$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ and $f(x) = -1$. This implies that the limit does not exist.

CHECKPoint Now try Exercise 31.

Example 7 Unbounded Behavior

Discuss the existence of the limit.

$$\lim_{x \to 0} \frac{1}{x^2}$$

Solution

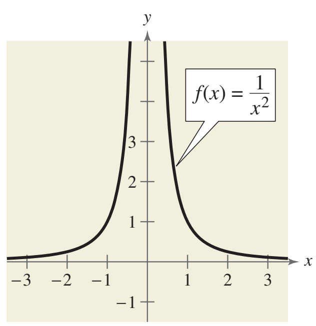

FIGURE 12.7

Let $f(x) = 1/x^2$. In Figure 12.7, note that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 if you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000, as follows.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a unique real number L as x approaches 0, you can conclude that the limit does not exist.

CHECKPoint Now try Exercise 33.

Example 8 Oscillating Behavior

Discuss the existence of the limit.

$$\lim_{x \to 0} \sin \frac{1}{x}$$

Solution

Let $f(x) = \sin(1/x)$. In Figure 12.8, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1. Therefore, the limit does not exist because no matter how close you are to 0, it is possible to choose values of x_1 and x_2 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as indicated in the table.

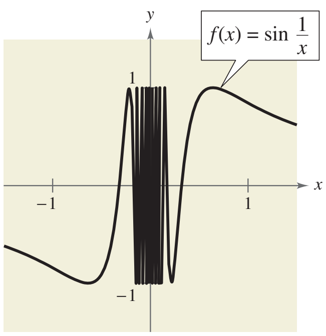

FIGURE 12.8

x	$-\frac{2}{\pi}$	$-\frac{2}{3\pi}$	$-\frac{2}{5\pi}$	0	$\frac{2}{5\pi}$	$\frac{2}{3\pi}$	$\frac{2}{\pi}$
$\sin \frac{1}{x}$	-1	1	-1	?	1	-1	1

CHECK*Point* Now try Exercise 35.

Examples 6, 7, and 8 show three of the most common types of behavior associated with the *nonexistence* of a limit.

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist if any of the following conditions are true.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c. — Example 6
2. $f(x)$ increases or decreases without bound as x approaches c. — Example 7
3. $f(x)$ oscillates between two fixed values as x approaches c. — Example 8

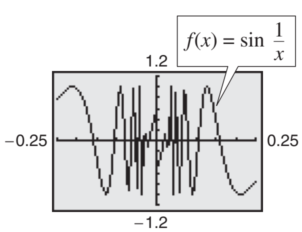

FIGURE 12.9

TECHNOLOGY

A graphing utility can help you discover the behavior of a function near the *x*-value at which you are trying to evaluate a limit. When you do this, however, you should realize that you can't always trust the graphs that graphing utilities display. For instance, if you use a graphing utility to graph the function in Example 8 over an interval containing 0, you will most likely obtain an incorrect graph, as shown in Figure 12.9. The reason that a graphing utility can't show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.

Properties of Limits and Direct Substitution

You have seen that sometimes the limit of $f(x)$ as $x \to c$ is simply $f(c)$, as shown in Example 2. In such cases, it is said that the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

There are many "well-behaved" functions, such as polynomial functions and rational functions with nonzero denominators, that have this property. Some of the basic ones are included in the following list.

Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$ — Limit of a constant function
2. $\lim_{x \to c} x = c$ — Limit of the identity function
3. $\lim_{x \to c} x^n = c^n$ — Limit of a power function
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$ — Limit of a radical function

For a proof of the limit of a power function, see Proofs in Mathematics on page 906.

Trigonometric functions can also be included in this list. For instance,

$$\lim_{x \to \pi} \sin x = \sin \pi = 0$$

and

$$\lim_{x \to 0} \cos x = \cos 0 = 1.$$

By combining the basic limits with the following operations, you can find limits for a wide variety of functions.

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$
2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Example 9 Direct Substitution and Properties of Limits

Find each limit.

a. $\lim_{x\to 4} x^2$ **b.** $\lim_{x\to 4} 5x$ **c.** $\lim_{x\to \pi} \frac{\tan x}{x}$

d. $\lim_{x\to 9} \sqrt{x}$ **e.** $\lim_{x\to \pi} (x \cos x)$ **f.** $\lim_{x\to 3} (x+4)^2$

Solution

You can use the properties of limits and direct substitution to evaluate each limit.

a. $\lim_{x\to 4} x^2 = (4)^2$

$= 16$

b. $\lim_{x\to 4} 5x = 5 \lim_{x\to 4} x$ Property 1

$= 5(4)$

$= 20$

c. $\lim_{x\to \pi} \frac{\tan x}{x} = \frac{\lim_{x\to\pi} \tan x}{\lim_{x\to\pi} x}$ Property 4

$= \frac{0}{\pi} = 0$

d. $\lim_{x\to 9} \sqrt{x} = \sqrt{9} = 3$

e. $\lim_{x\to \pi} (x \cos x) = \left(\lim_{x\to\pi} x\right)\left(\lim_{x\to\pi} \cos x\right)$ Property 3

$= \pi(\cos \pi)$

$= -\pi$

f. $\lim_{x\to 3} (x+4)^2 = \left[\left(\lim_{x\to 3} x\right) + \left(\lim_{x\to 3} 4\right)\right]^2$ Properties 2 and 5

$= (3+4)^2$

$= 7^2 = 49$

CHECKPoint Now try Exercise 47.

When evaluating limits, remember that there are several ways to solve most problems. Often, a problem can be solved *numerically*, *graphically*, or *algebraically*. The limits in Example 9 were found algebraically. You can verify the solutions numerically and/or graphically. For instance, to verify the limit in Example 9(a) numerically, create a table that shows values of x^2 for two sets of x-values—one set that approaches 4 from the left and one that approaches 4 from the right, as shown below. From the table, you can see that the limit as x approaches 4 is 16. Now, to verify the limit graphically, sketch the graph of $y = x^2$. From the graph shown in Figure 12.10, you can determine that the limit as x approaches 4 is 16.

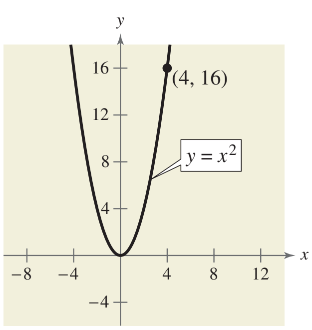

FIGURE 12.10

x	3.9	3.99	3.999	4.0	4.001	4.01	4.1
x^2	15.2100	15.9201	15.9920	?	16.0080	16.0801	16.8100

The results of using direct substitution to evaluate limits of polynomial and rational functions are summarized as follows.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

2. If r is a rational function given by $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

For a proof of the limit of a polynomial function, see Proofs in Mathematics on page 906.

Example 10 Evaluating Limits by Direct Substitution

Find each limit.

a. $\lim_{x \to -1} (x^2 + x - 6)$ **b.** $\lim_{x \to -1} \dfrac{x^2 + x - 6}{x + 3}$

Solution

The first function is a polynomial function and the second is a rational function (with a nonzero denominator at $x = -1$). So, you can evaluate the limits by direct substitution.

a. $$\lim_{x \to -1} (x^2 + x - 6) = (-1)^2 + (-1) - 6 = -6$$

b. $$\lim_{x \to -1} \frac{x^2 + x - 6}{x + 3} = \frac{(-1)^2 + (-1) - 6}{-1 + 3} = -\frac{6}{2} = -3$$

CHECK*Point* Now try Exercise 51.

Classroom Discussion

Graphs with Holes Sketch the graph of each function. Then find the limits of each function as x approaches 1 and as x approaches 2. What conclusions can you make?

a. $f(x) = x + 1$ b. $g(x) = \dfrac{x^2 - 1}{x - 1}$ c. $h(x) = \dfrac{x^3 - 2x^2 - x + 2}{x^2 - 3x + 2}$

Use a graphing utility to graph each function above. Does the graphing utility distinguish among the three graphs? Write a short explanation of your findings.