

10.7 POLAR COORDINATES

What you should learn

- Plot points on the polar coordinate system.
- Convert points from rectangular to polar form and vice versa.
- Convert equations from rectangular to polar form and vice versa.

Why you should learn it

Polar coordinates offer a different mathematical perspective on graphing. For instance, in Exercises 5–18 on page 781, you are asked to find multiple representations of polar coordinates.

Introduction

So far, you have been representing graphs of equations as collections of points (x, y) on the rectangular coordinate system, where x and y represent the directed distances from the coordinate axes to the point (x, y) . In this section, you will study a different system called the **polar coordinate system**.

To form the polar coordinate system in the plane, fix a point O , called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis**, as shown in Figure 10.59. Then each point P in the plane can be assigned **polar coordinates** (r, θ) as follows.

1. $r =$ directed distance from O to P
2. $\theta =$ directed angle, counterclockwise from polar axis to segment \overline{OP}

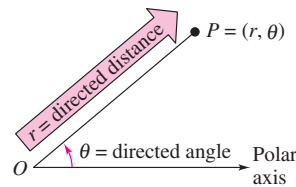


FIGURE 10.59

Example 1 Plotting Points on the Polar Coordinate System

- a. The point $(r, \theta) = (2, \pi/3)$ lies two units from the pole on the terminal side of the angle $\theta = \pi/3$, as shown in Figure 10.60.
- b. The point $(r, \theta) = (3, -\pi/6)$ lies three units from the pole on the terminal side of the angle $\theta = -\pi/6$, as shown in Figure 10.61.
- c. The point $(r, \theta) = (3, 11\pi/6)$ coincides with the point $(3, -\pi/6)$, as shown in Figure 10.62.

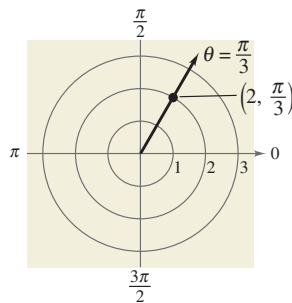


FIGURE 10.60

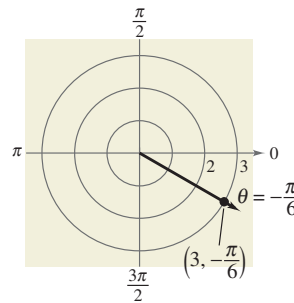


FIGURE 10.61

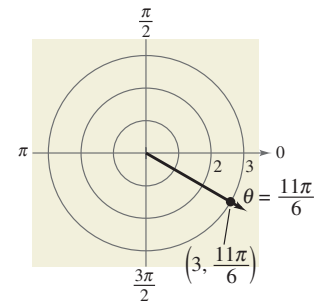


FIGURE 10.62

CheckPoint → Now try Exercise 7.

In rectangular coordinates, each point (x, y) has a unique representation. This is not true for polar coordinates. For instance, the coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point, as illustrated in Example 1. Another way to obtain multiple representations of a point is to use negative values for r . Because r is a *directed distance*, the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point. In general, the point (r, θ) can be represented as

$$(r, \theta) = (r, \theta \pm 2n\pi) \quad \text{or} \quad (r, \theta) = (-r, \theta \pm (2n + 1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

Example 2 Multiple Representations of Points

Plot the point $(3, -3\pi/4)$ and find three additional polar representations of this point, using $-2\pi < \theta < 2\pi$.

Solution

The point is shown in Figure 10.63. Three other representations are as follows.

$$\begin{aligned} (3, -\frac{3\pi}{4} + 2\pi) &= (3, \frac{5\pi}{4}) && \text{Add } 2\pi \text{ to } \theta. \\ (-3, -\frac{3\pi}{4} - \pi) &= (-3, -\frac{7\pi}{4}) && \text{Replace } r \text{ by } -r; \text{ subtract } \pi \text{ from } \theta. \\ (-3, -\frac{3\pi}{4} + \pi) &= (-3, \frac{\pi}{4}) && \text{Replace } r \text{ by } -r; \text{ add } \pi \text{ to } \theta. \end{aligned}$$

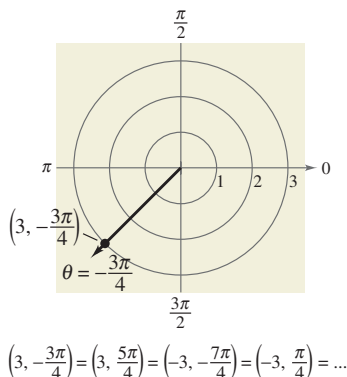


FIGURE 10.63

CheckPoint Now try Exercise 13.

Coordinate Conversion

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x -axis and the pole with the origin, as shown in Figure 10.64. Because (x, y) lies on a circle of radius r , it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$, the definitions of the trigonometric functions imply that

$$\tan \theta = \frac{y}{x}, \quad \cos \theta = \frac{x}{r}, \quad \text{and} \quad \sin \theta = \frac{y}{r}.$$

If $r < 0$, you can show that the same relationships hold.

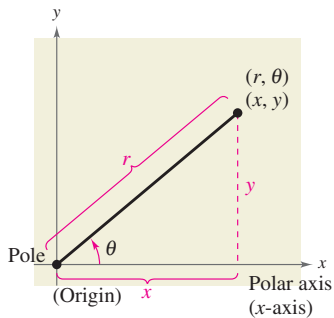


FIGURE 10.64

Coordinate Conversion

The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows.

Polar-to-Rectangular	Rectangular-to-Polar
$x = r \cos \theta$	$\tan \theta = \frac{y}{x}$
$y = r \sin \theta$	$r^2 = x^2 + y^2$

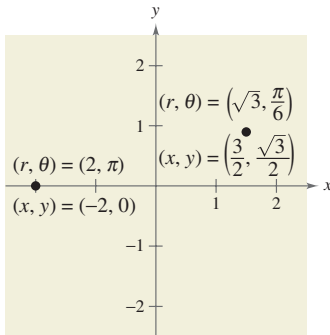


FIGURE 10.65

Example 3 Polar-to-Rectangular Conversion

Convert each point to rectangular coordinates.

- a. $(2, \pi)$ b. $(\sqrt{3}, \frac{\pi}{6})$

Solution

a. For the point $(r, \theta) = (2, \pi)$, you have the following.

$$x = r \cos \theta = 2 \cos \pi = -2$$

$$y = r \sin \theta = 2 \sin \pi = 0$$

The rectangular coordinates are $(x, y) = (-2, 0)$. (See Figure 10.65.)

b. For the point $(r, \theta) = (\sqrt{3}, \frac{\pi}{6})$, you have the following.

$$x = \sqrt{3} \cos \frac{\pi}{6} = \sqrt{3} \left(\frac{\sqrt{3}}{2} \right) = \frac{3}{2}$$

$$y = \sqrt{3} \sin \frac{\pi}{6} = \sqrt{3} \left(\frac{1}{2} \right) = \frac{\sqrt{3}}{2}$$

The rectangular coordinates are $(x, y) = (\frac{3}{2}, \frac{\sqrt{3}}{2})$.

CHECKPOINT Now try Exercise 23.

Example 4 Rectangular-to-Polar Conversion

Convert each point to polar coordinates.

- a. $(-1, 1)$ b. $(0, 2)$

Solution

a. For the second-quadrant point $(x, y) = (-1, 1)$, you have

$$\tan \theta = \frac{y}{x} = -1$$

$$\theta = \frac{3\pi}{4}$$

Because θ lies in the same quadrant as (x, y) , use positive r .

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

So, one set of polar coordinates is $(r, \theta) = (\sqrt{2}, 3\pi/4)$, as shown in Figure 10.66.

b. Because the point $(x, y) = (0, 2)$ lies on the positive y -axis, choose

$$\theta = \frac{\pi}{2} \quad \text{and} \quad r = 2.$$

This implies that one set of polar coordinates is $(r, \theta) = (2, \pi/2)$, as shown in Figure 10.67.

CHECKPOINT Now try Exercise 41.

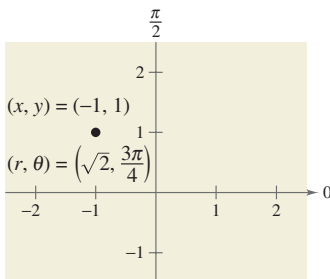


FIGURE 10.66

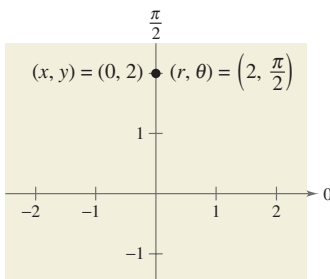


FIGURE 10.67

Equation Conversion

By comparing Examples 3 and 4, you can see that point conversion from the polar to the rectangular system is straightforward, whereas point conversion from the rectangular to the polar system is more involved. For equations, the opposite is true. To convert a rectangular equation to polar form, you simply replace x by $r \cos \theta$ and y by $r \sin \theta$. For instance, the rectangular equation $y = x^2$ can be written in polar form as follows.

$$\begin{aligned}
 y &= x^2 && \text{Rectangular equation} \\
 r \sin \theta &= (r \cos \theta)^2 && \text{Polar equation} \\
 r &= \sec \theta \tan \theta && \text{Simplest form}
 \end{aligned}$$

On the other hand, converting a polar equation to rectangular form requires considerable ingenuity.

Example 5 demonstrates several polar-to-rectangular conversions that enable you to sketch the graphs of some polar equations.

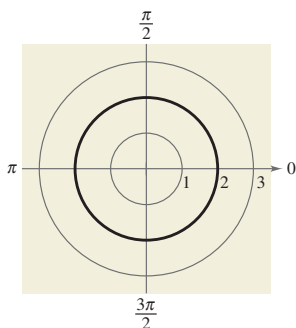


FIGURE 10.68

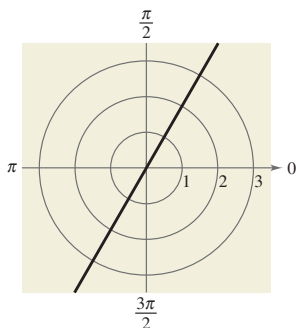


FIGURE 10.69

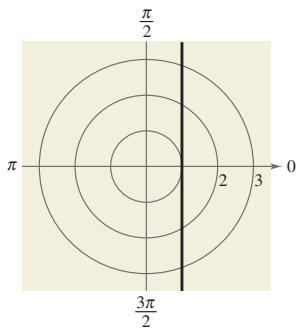


FIGURE 10.70

Example 5 Converting Polar Equations to Rectangular Form

Describe the graph of each polar equation and find the corresponding rectangular equation.

- a. $r = 2$ b. $\theta = \frac{\pi}{3}$ c. $r = \sec \theta$

Solution

- a. The graph of the polar equation $r = 2$ consists of all points that are two units from the pole. In other words, this graph is a circle centered at the origin with a radius of 2, as shown in Figure 10.68. You can confirm this by converting to rectangular form, using the relationship $r^2 = x^2 + y^2$.

$$\underbrace{r = 2}_{\text{Polar equation}} \quad \Rightarrow \quad r^2 = 2^2 \quad \Rightarrow \quad \underbrace{x^2 + y^2 = 2^2}_{\text{Rectangular equation}}$$

- b. The graph of the polar equation $\theta = \pi/3$ consists of all points on the line that makes an angle of $\pi/3$ with the positive polar axis, as shown in Figure 10.69. To convert to rectangular form, make use of the relationship $\tan \theta = y/x$.

$$\underbrace{\theta = \frac{\pi}{3}}_{\text{Polar equation}} \quad \Rightarrow \quad \tan \theta = \sqrt{3} \quad \Rightarrow \quad \underbrace{y = \sqrt{3}x}_{\text{Rectangular equation}}$$

- c. The graph of the polar equation $r = \sec \theta$ is not evident by simple inspection, so convert to rectangular form by using the relationship $r \cos \theta = x$.

$$\underbrace{r = \sec \theta}_{\text{Polar equation}} \quad \Rightarrow \quad r \cos \theta = 1 \quad \Rightarrow \quad \underbrace{x = 1}_{\text{Rectangular equation}}$$

Now you see that the graph is a vertical line, as shown in Figure 10.70.

CHECKPOINT Now try Exercise 109.