# 10.4 HYPERBOLAS

## What you should learn

- Write equations of hyperbolas in standard form.
- Find asymptotes of and graph hyperbolas.
- Use properties of hyperbolas to solve real-life problems.
- Classify conics from their general equations.

## Why you should learn it

Hyperbolas can be used to model and solve many types of real-life problems. For instance, in Exercise 54 on page 759, hyperbolas are used in long distance radio navigation for aircraft and ships.

U.S. Navy, William Lipski/AP Photo

## Introduction

The third type of conic is called a **hyperbola.** The definition of a hyperbola is similar to that of an ellipse. The difference is that for an ellipse the *sum* of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola the *difference* of the distances between the foci and a point on the hyperbola is fixed.

### Definition of Hyperbola

A **hyperbola** is the set of all points $(x, y)$ in a plane, the difference of whose distances from two distinct fixed points **(foci)** is a positive constant. See Figure 10.30.

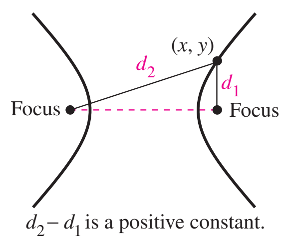



FIGURE 10.30

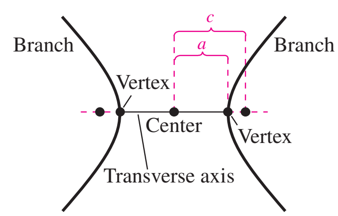



FIGURE 10.31

The graph of a hyperbola has two disconnected **branches.** The line through the two foci intersects the hyperbola at its two **vertices.** The line segment connecting the vertices is the **transverse axis,** and the midpoint of the transverse axis is the **center** of the hyperbola. See Figure 10.31. The development of the standard form of the equation of a hyperbola is similar to that of an ellipse. Note in the definition below that $a$, $b$, and $c$ are related differently for hyperbolas than for ellipses.

### Standard Equation of a Hyperbola

The **standard form of the equation of a hyperbola** with center $(h, k)$ is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

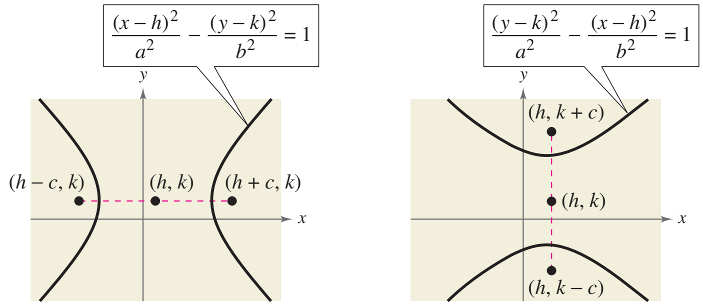
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad \text{Transverse axis is vertical.}$$

The vertices are $a$ units from the center, and the foci are $c$ units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{Transverse axis is vertical.}$$

Figure 10.32 shows both the horizontal and vertical orientations for a hyperbola.




Transverse axis is horizontal. Transverse axis is vertical.

FIGURE 10.32

## Example 1 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci $(-1, 2)$ and $(5, 2)$ and vertices $(0, 2)$ and $(4, 2)$.

### Solution

By the Midpoint Formula, the center of the hyperbola occurs at the point $(2, 2)$. Furthermore, $c = 5 - 2 = 3$ and $a = 4 - 2 = 2$, and it follows that

$$b = \sqrt{c^2 - a^2} = \sqrt{3^2 - 2^2} = \sqrt{9 - 4} = \sqrt{5}.$$

So, the hyperbola has a horizontal transverse axis and the standard form of the equation is

$$\frac{(x - 2)^2}{2^2} - \frac{(y - 2)^2}{\left(\sqrt{5}\right)^2} = 1.$$ See Figure 10.33.

This equation simplifies to

$$\frac{(x - 2)^2}{4} - \frac{(y - 2)^2}{5} = 1.$$

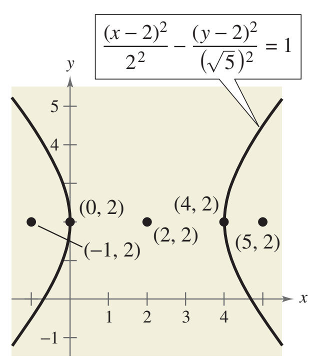



FIGURE 10.33

CHECKPoint Now try Exercise 35.

**Study Tip**

When finding the standard form of the equation of any conic, it is helpful to sketch a graph of the conic with the given characteristics.

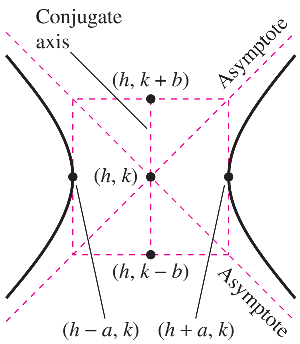



FIGURE 10.34

## Asymptotes of a Hyperbola

Each hyperbola has two **asymptotes** that intersect at the center of the hyperbola, as shown in Figure 10.34. The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at $(h, k)$. The line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ $[$or $(h + b, k)$ and $(h - b, k)]$ is the **conjugate axis** of the hyperbola.

### Asymptotes of a Hyperbola

The equations of the asymptotes of a hyperbola are

$$y = k \pm \frac{b}{a}(x - h) \qquad \text{Transverse axis is horizontal.}$$

$$y = k \pm \frac{a}{b}(x - h). \qquad \text{Transverse axis is vertical.}$$

### Example 2 Using Asymptotes to Sketch a Hyperbola

Sketch the hyperbola whose equation is $4x^2 - y^2 = 16$.

#### Algebraic Solution

Divide each side of the original equation by 16, and rewrite the equation in standard form.

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1 \qquad \text{Write in standard form.}$$

From this, you can conclude that $a = 2$, $b = 4$, and the transverse axis is horizontal. So, the vertices occur at $(-2, 0)$ and $(2, 0)$, and the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$. Using these four points, you are able to sketch the rectangle shown in Figure 10.35. Now, from $c^2 = a^2 + b^2$, you have $c = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. So, the foci of the hyperbola are $(-2\sqrt{5}, 0)$ and $(2\sqrt{5}, 0)$. Finally, by drawing the asymptotes through the corners of this rectangle, you can complete the sketch shown in Figure 10.36. Note that the asymptotes are $y = 2x$ and $y = -2x$.

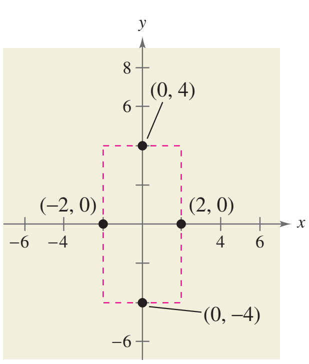



FIGURE 10.35

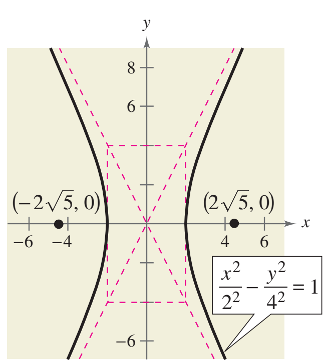



FIGURE 10.36

CHECKPoint Now try Exercise 11.

#### Graphical Solution

Solve the equation of the hyperbola for $y$ as follows.

$$4x^2 - y^2 = 16$$

$$4x^2 - 16 = y^2$$

$$\pm\sqrt{4x^2 - 16} = y$$

Then use a graphing utility to graph $y_1 = \sqrt{4x^2 - 16}$ and $y_2 = -\sqrt{4x^2 - 16}$ in the same viewing window. Be sure to use a square setting. From the graph in Figure 10.37, you can see that the transverse axis is horizontal. You can use the *zoom* and *trace* features to approximate the vertices to be $(-2, 0)$ and $(2, 0)$.

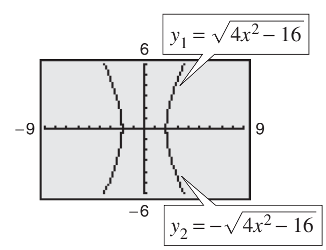



FIGURE 10.37

## Example 3 Finding the Asymptotes of a Hyperbola

Sketch the hyperbola given by $4x^2 - 3y^2 + 8x + 16 = 0$ and find the equations of its asymptotes and the foci.

### Solution

$$4x^2 - 3y^2 + 8x + 16 = 0 \qquad \text{Write original equation.}$$

$$(4x^2 + 8x) - 3y^2 = -16 \qquad \text{Group terms.}$$

$$4(x^2 + 2x) - 3y^2 = -16 \qquad \text{Factor 4 from } x\text{-terms.}$$

$$4(x^2 + 2x + 1) - 3y^2 = -16 + 4 \qquad \text{Add 4 to each side.}$$

$$4(x + 1)^2 - 3y^2 = -12 \qquad \text{Write in completed square form.}$$

$$-\frac{(x + 1)^2}{3} + \frac{y^2}{4} = 1 \qquad \text{Divide each side by } -12.$$

$$\frac{y^2}{2^2} - \frac{(x + 1)^2}{(\sqrt{3})^2} = 1 \qquad \text{Write in standard form.}$$

From this equation you can conclude that the hyperbola has a vertical transverse axis, centered at $(-1, 0)$, has vertices $(-1, 2)$ and $(-1, -2)$, and has a conjugate axis with endpoints $\left(-1 - \sqrt{3}, 0\right)$ and $\left(-1 + \sqrt{3}, 0\right)$. To sketch the hyperbola, draw a rectangle through these four points. The asymptotes are the lines passing through the corners of the rectangle. Using $a = 2$ and $b = \sqrt{3}$, you can conclude that the equations of the asymptotes are

$$y = \frac{2}{\sqrt{3}}(x + 1) \quad \text{and} \quad y = -\frac{2}{\sqrt{3}}(x + 1).$$

Finally, you can determine the foci by using the equation $c^2 = a^2 + b^2$. So, you have $c = \sqrt{2^2 + \left(\sqrt{3}\right)^2} = \sqrt{7}$, and the foci are $\left(-1, \sqrt{7}\right)$ and $\left(-1, -\sqrt{7}\right)$. The hyperbola is shown in Figure 10.38.

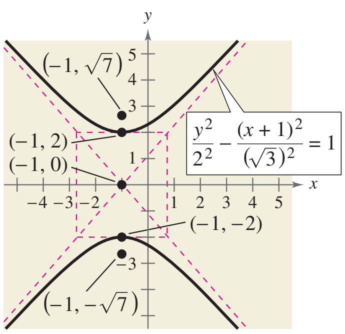



FIGURE 10.38

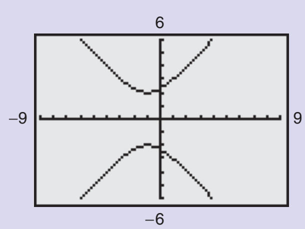
CHECKPoint Now try Exercise 19.

### TECHNOLOGY

You can use a graphing utility to graph a hyperbola by graphing the upper and lower portions in the same viewing window. For instance, to graph the hyperbola in Example 3, first solve for $y$ to get

$$y_1 = 2\sqrt{1 + \frac{(x + 1)^2}{3}} \quad \text{and} \quad y_2 = -2\sqrt{1 + \frac{(x + 1)^2}{3}}.$$

Use a viewing window in which $-9 \le x \le 9$ and $-6 \le y \le 6$. You should obtain the graph shown below. Notice that the graphing utility does not draw the asymptotes. However, if you trace along the branches, you will see that the values of the hyperbola approach the asymptotes.



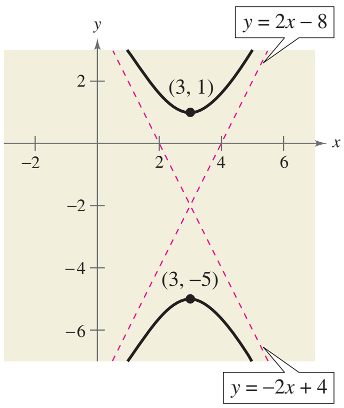



FIGURE 10.39

## Example 4 Using Asymptotes to Find the Standard Equation

Find the standard form of the equation of the hyperbola having vertices $(3, -5)$ and $(3, 1)$ and having asymptotes

$$y = 2x - 8 \quad \text{and} \quad y = -2x + 4$$

as shown in Figure 10.39.

### Solution

By the Midpoint Formula, the center of the hyperbola is $(3, -2)$. Furthermore, the hyperbola has a vertical transverse axis with $a = 3$. From the original equations, you can determine the slopes of the asymptotes to be

$$m_1 = 2 = \frac{a}{b} \quad \text{and} \quad m_2 = -2 = -\frac{a}{b}$$

and, because $a = 3$, you can conclude

$$2 = \frac{a}{b} \quad \Rightarrow \quad 2 = \frac{3}{b} \quad \Rightarrow \quad b = \frac{3}{2}.$$

So, the standard form of the equation is

$$\frac{(y+2)^2}{3^2} - \frac{(x-3)^2}{\left(\frac{3}{2}\right)^2} = 1.$$

**CHECKPoint** Now try Exercise 43.

As with ellipses, the *eccentricity* of a hyperbola is

$$e = \frac{c}{a} \qquad \text{Eccentricity}$$

and because $c > a$, it follows that $e > 1$. If the eccentricity is large, the branches of the hyperbola are nearly flat, as shown in Figure 10.40. If the eccentricity is close to 1, the branches of the hyperbola are more narrow, as shown in Figure 10.41.

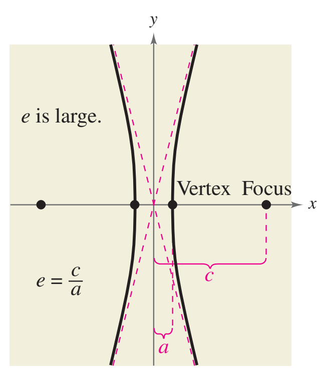



FIGURE 10.40

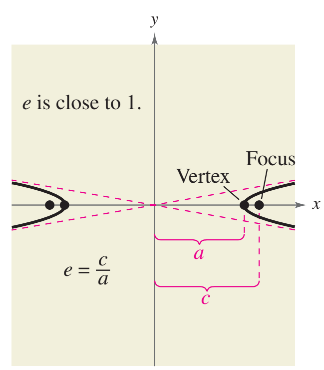



FIGURE 10.41

## Applications

The following application was developed during World War II. It shows how the properties of hyperbolas can be used in radar and other detection systems.

### Example 5 An Application Involving Hyperbolas

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where did the explosion occur? (Assume sound travels at 1100 feet per second.)

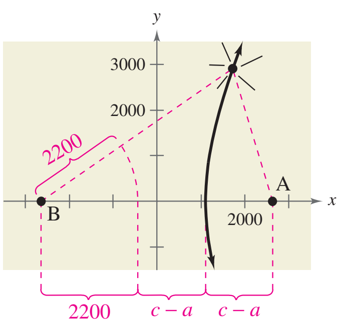



FIGURE 10.42

#### Solution

Assuming sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from B than from A, as shown in Figure 10.42. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$c = \frac{5280}{2} = 2640$$

and

$$a = \frac{2200}{2} = 1100.$$

So, $b^2 = c^2 - a^2 = 2640^2 - 1100^2 = 5{,}759{,}600$, and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$

CHECKPoint Now try Exercise 53.

Another interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure 10.43. Undoubtedly, there have been many comets with parabolic or hyperbolic orbits that were not identified. We only get to see such comets *once*. Comets with elliptical orbits, such as Halley's comet, are the only ones that remain in our solar system.

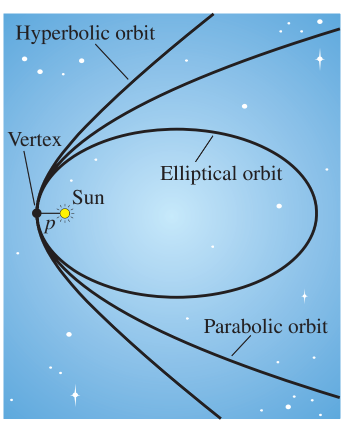



FIGURE 10.43

If $p$ is the distance between the vertex and the focus (in meters), and $v$ is the velocity of the comet at the vertex (in meters per second), then the type of orbit is determined as follows.

1. Ellipse: $v < \sqrt{2GM/p}$
2. Parabola: $v = \sqrt{2GM/p}$
3. Hyperbola: $v > \sqrt{2GM/p}$

In each of these relations, $M = 1.989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

## General Equations of Conics

### Classifying a Conic from Its General Equation

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is one of the following.

1. *Circle:* $A = C$
2. *Parabola:* $AC = 0$ — $A = 0$ or $C = 0$, but not both.
3. *Ellipse:* $AC > 0$ — $A$ and $C$ have like signs.
4. *Hyperbola:* $AC < 0$ — $A$ and $C$ have unlike signs.

The test above is valid *if* the graph is a conic. The test does not apply to equations such as $x^2 + y^2 = -1$, whose graph is not a conic.

### Example 6 Classifying Conics from General Equations

Classify the graph of each equation.

**a.** $4x^2 - 9x + y - 5 = 0$

**b.** $4x^2 - y^2 + 8x - 6y + 4 = 0$

**c.** $2x^2 + 4y^2 - 4x + 12y = 0$

**d.** $2x^2 + 2y^2 - 8x + 12y + 2 = 0$

#### Solution

**a.** For the equation $4x^2 - 9x + y - 5 = 0$, you have

$$AC = 4(0) = 0. \qquad \text{Parabola}$$

So, the graph is a parabola.

**b.** For the equation $4x^2 - y^2 + 8x - 6y + 4 = 0$, you have

$$AC = 4(-1) < 0. \qquad \text{Hyperbola}$$

So, the graph is a hyperbola.

**c.** For the equation $2x^2 + 4y^2 - 4x + 12y = 0$, you have

$$AC = 2(4) > 0. \qquad \text{Ellipse}$$

So, the graph is an ellipse.

**d.** For the equation $2x^2 + 2y^2 - 8x + 12y + 2 = 0$, you have

$$A = C = 2. \qquad \text{Circle}$$

So, the graph is a circle.

CHECKPoint Now try Exercise 61.

### Classroom Discussion

**Sketching Conics** Sketch each of the conics described in Example 6. Write a paragraph describing the procedures that allow you to sketch the conics efficiently.

### HISTORICAL NOTE


Caroline Herschel (1750–1848) was the first woman to be credited with detecting a new comet. During her long life, this English astronomer discovered a total of eight new comets.