10.3 Ellipses

What you should learn

- Write equations of ellipses in standard form and graph ellipses.
- Use properties of ellipses to model and solve real-life problems.
- Find eccentricities of ellipses.

Why you should learn it

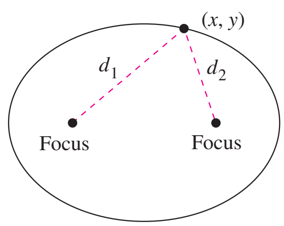
Ellipses can be used to model and solve many types of real-life problems. For instance, in Exercise 65 on page 749, an ellipse is used to model the orbit of Halley's comet.

Harvard College Observatory/Photo Researchers, Inc.

Introduction

The second type of conic is called an **ellipse,** and is defined as follows.

> **Definition of Ellipse**
>
> An **ellipse** is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points **(foci)** is constant. See Figure 10.19.



$d_1 + d_2$ is constant.

FIGURE 10.19

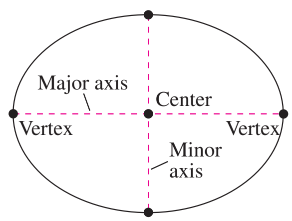


FIGURE 10.20

The line through the foci intersects the ellipse at two points called **vertices.** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis** of the ellipse. See Figure 10.20.

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 10.21. If the ends of a fixed length of string are fastened to the thumbtacks and the string is *drawn taut* with a pencil, the path traced by the pencil will be an ellipse.

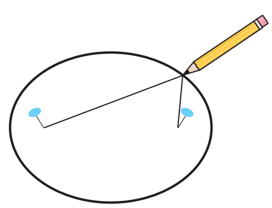


FIGURE 10.21

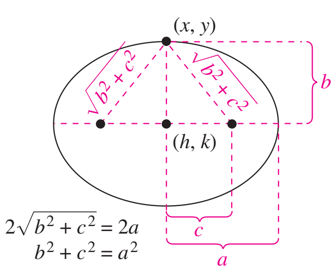


FIGURE 10.22

To derive the standard form of the equation of an ellipse, consider the ellipse in Figure 10.22 with the following points: center, (h, k); vertices, $(h \pm a, k)$; foci, $(h \pm c, k)$. Note that the center is the midpoint of the segment joining the foci. The sum of the distances from any point on the ellipse to the two foci is constant. Using a vertex point, this constant sum is

$$(a + c) + (a - c) = 2a \qquad \text{Length of major axis}$$

or simply the length of the major axis. Now, if you let (x, y) be *any* point on the ellipse, the sum of the distances between (x, y) and the two foci must also be $2a$.

That is,

$$\sqrt{[x-(h-c)]^2+(y-k)^2}+\sqrt{[x-(h+c)]^2+(y-k)^2}=2a$$

which, after expanding and regrouping, reduces to

$$(a^2-c^2)(x-h)^2+a^2(y-k)^2=a^2(a^2-c^2).$$

Finally, in Figure 10.22, you can see that

$$b^2=a^2-c^2$$

which implies that the equation of the ellipse is

$$b^2(x-h)^2+a^2(y-k)^2=a^2b^2$$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

You would obtain a similar equation in the derivation by starting with a vertical major axis. Both results are summarized as follows.

Study Tip

Consider the equation of the ellipse

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

If you let $a=b$, then the equation can be rewritten as

$$(x-h)^2+(y-k)^2=a^2$$

which is the standard form of the equation of a circle with radius $r=a$ (see Section 1.2). Geometrically, when $a=b$ for an ellipse, the major and minor axes are of equal length, and so the graph is a circle.

Standard Equation of an Ellipse

The **standard form of the equation of an ellipse,** with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectively, where $0<b<a$, is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

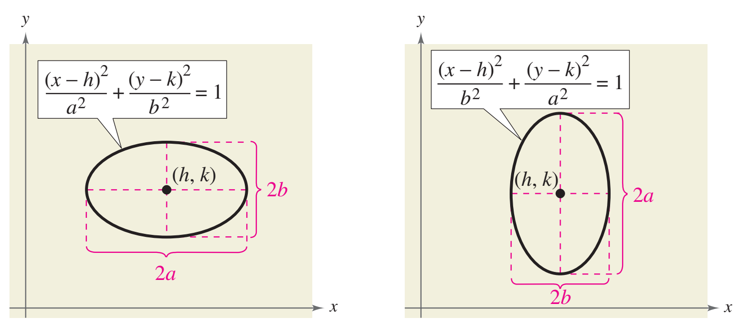
$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with $c^2=a^2-b^2$. If the center is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad \text{Major axis is vertical.}$$

Figure 10.23 shows both the horizontal and vertical orientations for an ellipse.



Major axis is horizontal. Major axis is vertical.

FIGURE 10.23

Example 1 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse having foci at $(0, 1)$ and $(4, 1)$ and a major axis of length 6, as shown in Figure 10.24.

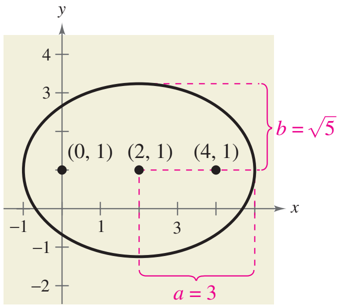


FIGURE 10.24

Solution

Because the foci occur at $(0, 1)$ and $(4, 1)$, the center of the ellipse is $(2, 1)$ and the distance from the center to one of the foci is $c = 2$. Because $2a = 6$, you know that $a = 3$. Now, from $c^2 = a^2 - b^2$, you have

$$b = \sqrt{a^2 - c^2} = \sqrt{3^2 - 2^2} = \sqrt{5}.$$

Because the major axis is horizontal, the standard equation is

$$\frac{(x-2)^2}{3^2} + \frac{(y-1)^2}{\left(\sqrt{5}\right)^2} = 1.$$

This equation simplifies to

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1.$$

CHECK*Point* Now try Exercise 23.

Example 2 Sketching an Ellipse

Sketch the ellipse given by $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

Solution

Begin by writing the original equation in standard form. In the fourth step, note that 9 and 4 are added to *both* sides of the equation when completing the squares.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0 \qquad \text{Write original equation.}$$

$$\left(x^2 + 6x + \square\right) + \left(4y^2 - 8y + \square\right) = -9 \qquad \text{Group terms.}$$

$$\left(x^2 + 6x + \square\right) + 4\left(y^2 - 2y + \square\right) = -9 \qquad \text{Factor 4 out of } y\text{-terms.}$$

$$\left(x^2 + 6x + 9\right) + 4\left(y^2 - 2y + 1\right) = -9 + 9 + 4(1)$$

$$(x+3)^2 + 4(y-1)^2 = 4 \qquad \text{Write in completed square form.}$$

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{1} = 1 \qquad \text{Divide each side by 4.}$$

$$\frac{(x+3)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1 \qquad \text{Write in standard form.}$$

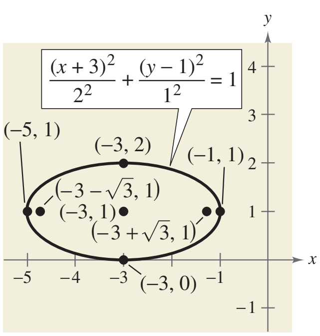


FIGURE 10.25

From this standard form, it follows that the center is $(h, k) = (-3, 1)$. Because the denominator of the x-term is $a^2 = 2^2$, the endpoints of the major axis lie two units to the right and left of the center. Similarly, because the denominator of the y-term is $b^2 = 1^2$, the endpoints of the minor axis lie one unit up and down from the center. Now, from $c^2 = a^2 - b^2$, you have $c = \sqrt{2^2 - 1^2} = \sqrt{3}$. So, the foci of the ellipse are $\left(-3 - \sqrt{3}, 1\right)$ and $\left(-3 + \sqrt{3}, 1\right)$. The ellipse is shown in Figure 10.25.

CHECK*Point* Now try Exercise 47.

Example 3 Analyzing an Ellipse

Find the center, vertices, and foci of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$.

Solution

By completing the square, you can write the original equation in standard form.

$$4x^2 + y^2 - 8x + 4y - 8 = 0 \qquad \text{Write original equation.}$$

$$(4x^2 - 8x + \square) + (y^2 + 4y + \square) = 8 \qquad \text{Group terms.}$$

$$4(x^2 - 2x + \square) + (y^2 + 4y + \square) = 8 \qquad \text{Factor 4 out of } x\text{-terms.}$$

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4(1) + 4$$

$$4(x - 1)^2 + (y + 2)^2 = 16 \qquad \text{Write in completed square form.}$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1 \qquad \text{Divide each side by 16.}$$

$$\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{4^2} = 1 \qquad \text{Write in standard form.}$$

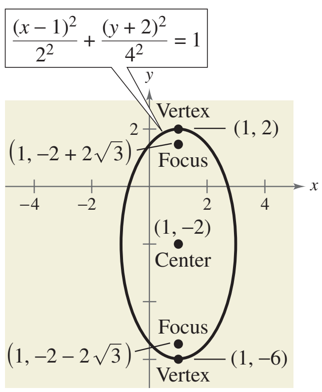


FIGURE 10.26

The major axis is vertical, where $h = 1$, $k = -2$, $a = 4$, $b = 2$, and

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}.$$

So, you have the following.

Center: $(1, -2)$ Vertices: $(1, -6)$ Foci: $(1, -2 - 2\sqrt{3})$

$(1, 2)$ $(1, -2 + 2\sqrt{3})$

The graph of the ellipse is shown in Figure 10.26.

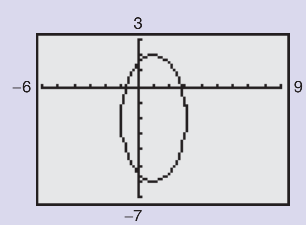
CHECK*Point* Now try Exercise 51.

TECHNOLOGY

You can use a graphing utility to graph an ellipse by graphing the upper and lower portions in the same viewing window. For instance, to graph the ellipse in Example 3, first solve for y to get

$$y_1 = -2 + 4\sqrt{1 - \frac{(x-1)^2}{4}} \quad \text{and} \quad y_2 = -2 - 4\sqrt{1 - \frac{(x-1)^2}{4}}.$$

Use a viewing window in which $-6 \le x \le 9$ and $-7 \le y \le 3$. You should obtain the graph shown below.



Application

Ellipses have many practical and aesthetic uses. For instance, machine gears, supporting arches, and acoustic designs often involve elliptical shapes. The orbits of satellites and planets are also ellipses. Example 4 investigates the elliptical orbit of the moon about Earth.

Example 4 An Application Involving an Elliptical Orbit

The moon travels about Earth in an elliptical orbit with Earth at one focus, as shown in Figure 10.27. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the greatest and smallest distances (the *apogee* and *perigee*, respectively) from Earth's center to the moon's center.

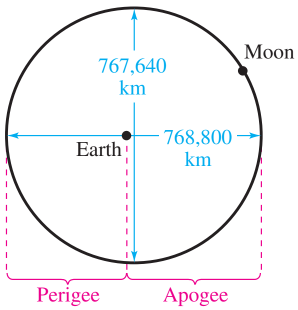


FIGURE 10.27

Solution

Because $2a = 768{,}800$ and $2b = 767{,}640$, you have

$$a = 384{,}400 \text{ and } b = 383{,}820$$

which implies that

$$\begin{aligned} c &= \sqrt{a^2 - b^2} \\ &= \sqrt{384{,}400^2 - 383{,}820^2} \\ &\approx 21{,}108. \end{aligned}$$

So, the greatest distance between the center of Earth and the center of the moon is

$$a + c \approx 384{,}400 + 21{,}108 = 405{,}508 \text{ kilometers}$$

and the smallest distance is

$$a - c \approx 384{,}400 - 21{,}108 = 363{,}292 \text{ kilometers}.$$

CHECKPoint Now try Exercise 65.

WARNING / CAUTION

Note in Example 4 and Figure 10.27 that Earth *is not* the center of the moon's orbit.

Eccentricity

One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity.**

Definition of Eccentricity

The **eccentricity** e of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$

Note that $0 < e < 1$ for *every* ellipse.

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is small, as shown in Figure 10.28. On the other hand, for an elongated ellipse, the foci are close to the vertices and the ratio c/a is close to 1, as shown in Figure 10.29.

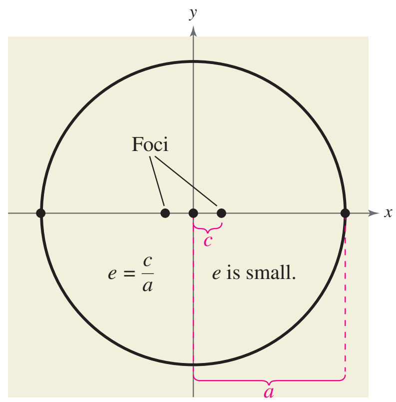


FIGURE 10.28

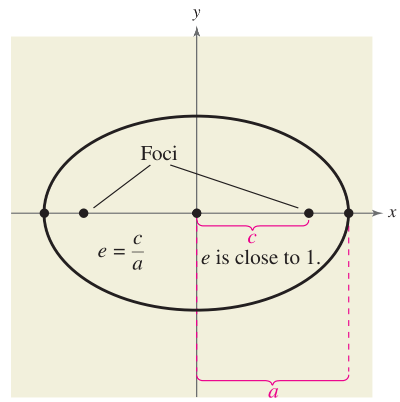
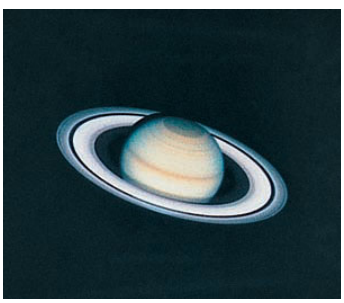


FIGURE 10.29



NASA

The time it takes Saturn to orbit the sun is about 29.4 Earth years.

The orbit of the moon has an eccentricity of $e \approx 0.0549$, and the eccentricities of the eight planetary orbits are as follows.

Mercury:	$e \approx 0.2056$	Jupiter:	$e \approx 0.0484$
Venus:	$e \approx 0.0068$	Saturn:	$e \approx 0.0542$
Earth:	$e \approx 0.0167$	Uranus:	$e \approx 0.0472$
Mars:	$e \approx 0.0934$	Neptune:	$e \approx 0.0086$

Classroom Discussion

Ellipses and Circles

a. Show that the equation of an ellipse can be written as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

b. For the equation in part (a), let $a = 4$, $h = 1$, and $k = 2$, and use a graphing utility to graph the ellipse for $e = 0.95$, $e = 0.75$, $e = 0.5$, $e = 0.25$, and $e = 0.1$. Discuss the changes in the shape of the ellipse as e approaches 0.

c. Make a conjecture about the shape of the graph in part (b) when $e = 0$. What is the equation of this ellipse? What is another name for an ellipse with an eccentricity of 0?