

4.7 INVERSE TRIGONOMETRIC FUNCTIONS

What you should learn

- Evaluate and graph the inverse sine function.
- Evaluate and graph the other inverse trigonometric functions.
- Evaluate and graph the compositions of trigonometric functions.

Why you should learn it

You can use inverse trigonometric functions to model and solve real-life problems. For instance, in Exercise 106 on page 349, an inverse trigonometric function can be used to model the angle of elevation from a television camera to a space shuttle launch.



Study Tip

When evaluating the inverse sine function, it helps to remember the phrase “the arcsine of x is the angle (or number) whose sine is x .”

Inverse Sine Function

Recall from Section 1.9 that, for a function to have an inverse function, it must be one-to-one—that is, it must pass the Horizontal Line Test. From Figure 4.71, you can see that $y = \sin x$ does not pass the test because different values of x yield the same y -value.

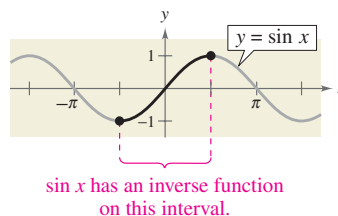


FIGURE 4.71

However, if you restrict the domain to the interval $-\pi/2 \leq x \leq \pi/2$ (corresponding to the black portion of the graph in Figure 4.71), the following properties hold.

1. On the interval $[-\pi/2, \pi/2]$, the function $y = \sin x$ is increasing.
2. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range of values, $-1 \leq \sin x \leq 1$.
3. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ is one-to-one.

So, on the restricted domain $-\pi/2 \leq x \leq \pi/2$, $y = \sin x$ has a unique inverse function called the **inverse sine function**. It is denoted by

$$y = \arcsin x \quad \text{or} \quad y = \sin^{-1} x.$$

The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$. The arcsin x notation (read as “the arcsine of x ”) comes from the association of a central angle with its intercepted *arc length* on a unit circle. So, arcsin x means the angle (or arc) whose sine is x . Both notations, arcsin x and $\sin^{-1} x$, are commonly used in mathematics, so remember that $\sin^{-1} x$ denotes the *inverse* sine function rather than $1/\sin x$. The values of arcsin x lie in the interval $-\pi/2 \leq \arcsin x \leq \pi/2$. The graph of $y = \arcsin x$ is shown in Example 2.

Definition of Inverse Sine Function

The **inverse sine function** is defined by

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$. The domain of $y = \arcsin x$ is $[-1, 1]$, and the range is $[-\pi/2, \pi/2]$.

Study Tip

As with the trigonometric functions, much of the work with the inverse trigonometric functions can be done by *exact* calculations rather than by calculator approximations. Exact calculations help to increase your understanding of the inverse functions by relating them to the right triangle definitions of the trigonometric functions.

Example 1 Evaluating the Inverse Sine Function

If possible, find the exact value.

- a. $\arcsin\left(-\frac{1}{2}\right)$ b. $\sin^{-1}\frac{\sqrt{3}}{2}$ c. $\sin^{-1}2$

Solution

- a. Because $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, it follows that

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}. \quad \text{Angle whose sine is } -\frac{1}{2}$$

- b. Because $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, it follows that

$$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}. \quad \text{Angle whose sine is } \frac{\sqrt{3}}{2}$$

- c. It is not possible to evaluate $y = \sin^{-1}x$ when $x = 2$ because there is no angle whose sine is 2. Remember that the domain of the inverse sine function is $[-1, 1]$.

CHECKPoint Now try Exercise 5.

Example 2 Graphing the Arcsine Function

Sketch a graph of

$$y = \arcsin x.$$

Solution

By definition, the equations $y = \arcsin x$ and $\sin y = x$ are equivalent for $-\pi/2 \leq y \leq \pi/2$. So, their graphs are the same. From the interval $[-\pi/2, \pi/2]$, you can assign values to y in the second equation to make a table of values. Then plot the points and draw a smooth curve through the points.

y	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x = \sin y$	-1	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1

The resulting graph for $y = \arcsin x$ is shown in Figure 4.72. Note that it is the reflection (in the line $y = x$) of the black portion of the graph in Figure 4.71. Be sure you see that Figure 4.72 shows the *entire* graph of the inverse sine function. Remember that the domain of $y = \arcsin x$ is the closed interval $[-1, 1]$ and the range is the closed interval $[-\pi/2, \pi/2]$.

CHECKPoint Now try Exercise 21.

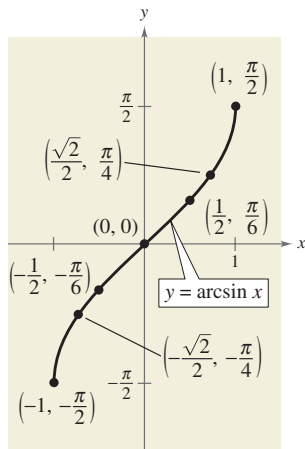


FIGURE 4.72

Other Inverse Trigonometric Functions

The cosine function is decreasing and one-to-one on the interval $0 \leq x \leq \pi$, as shown in Figure 4.73.

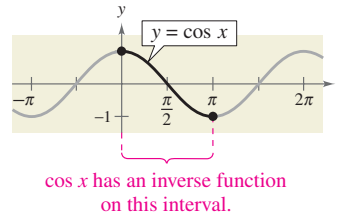


FIGURE 4.73

Consequently, on this interval the cosine function has an inverse function—the **inverse cosine function**—denoted by

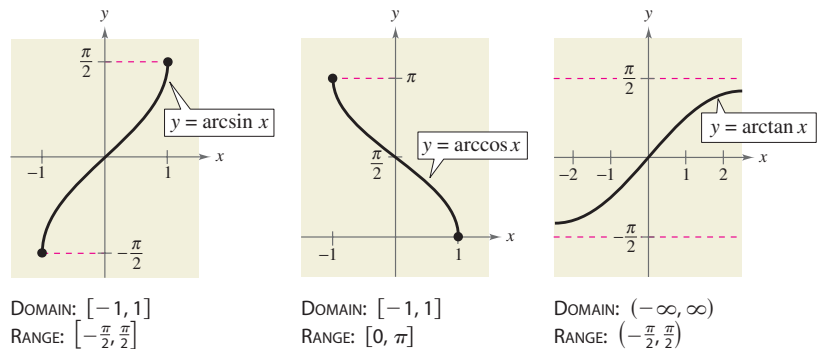
$$y = \arccos x \quad \text{or} \quad y = \cos^{-1} x.$$

Similarly, you can define an **inverse tangent function** by restricting the domain of $y = \tan x$ to the interval $(-\pi/2, \pi/2)$. The following list summarizes the definitions of the three most common inverse trigonometric functions. The remaining three are defined in Exercises 115–117.

Definitions of the Inverse Trigonometric Functions

Function	Domain	Range
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

The graphs of these three inverse trigonometric functions are shown in Figure 4.74.



DOMAIN: $[-1, 1]$
 RANGE: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
 FIGURE 4.74

DOMAIN: $[-1, 1]$
 RANGE: $[0, \pi]$

DOMAIN: $(-\infty, \infty)$
 RANGE: $(-\frac{\pi}{2}, \frac{\pi}{2})$

Example 3 Evaluating Inverse Trigonometric Functions

Find the exact value.

- a. $\arccos \frac{\sqrt{2}}{2}$ b. $\cos^{-1}(-1)$
 c. $\arctan 0$ d. $\tan^{-1}(-1)$

Solutiona. Because $\cos(\pi/4) = \sqrt{2}/2$, and $\pi/4$ lies in $[0, \pi]$, it follows that

$$\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}. \quad \text{Angle whose cosine is } \sqrt{2}/2$$

b. Because $\cos \pi = -1$, and π lies in $[0, \pi]$, it follows that

$$\cos^{-1}(-1) = \pi. \quad \text{Angle whose cosine is } -1$$

c. Because $\tan 0 = 0$, and 0 lies in $(-\pi/2, \pi/2)$, it follows that

$$\arctan 0 = 0. \quad \text{Angle whose tangent is } 0$$

d. Because $\tan(-\pi/4) = -1$, and $-\pi/4$ lies in $(-\pi/2, \pi/2)$, it follows that

$$\tan^{-1}(-1) = -\frac{\pi}{4}. \quad \text{Angle whose tangent is } -1$$

CHECKPOINT Now try Exercise 15.**Example 4** Calculators and Inverse Trigonometric Functions

Use a calculator to approximate the value (if possible).

- a. $\arctan(-8.45)$
 b. $\sin^{-1} 0.2447$
 c. $\arccos 2$

Solution

Function	Mode	Calculator Keystrokes
a. $\arctan(-8.45)$	Radian	$\boxed{\text{TAN}^{-1}} \boxed{(-)} \boxed{8.45} \boxed{)} \boxed{\text{ENTER}}$

From the display, it follows that $\arctan(-8.45) \approx -1.453001$.

b. $\sin^{-1} 0.2447$	Radian	$\boxed{\text{SIN}^{-1}} \boxed{)} \boxed{0.2447} \boxed{)} \boxed{\text{ENTER}}$
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From the display, it follows that $\sin^{-1} 0.2447 \approx 0.2472103$.

c. $\arccos 2$	Radian	$\boxed{\text{COS}^{-1}} \boxed{)} \boxed{2} \boxed{)} \boxed{\text{ENTER}}$
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In *real number* mode, the calculator should display an *error message* because the domain of the inverse cosine function is $[-1, 1]$.**CHECKPOINT** Now try Exercise 29.**WARNING / CAUTION**Remember that the domain of the inverse sine function and the inverse cosine function is $[-1, 1]$, as indicated in Example 4(c).In Example 4, if you had set the calculator to *degree* mode, the displays would have been in degrees rather than radians. This convention is peculiar to calculators. By definition, the values of inverse trigonometric functions are *always in radians*.

Algebra Help

You can review the composition of functions in Section 1.8.

Compositions of Functions

Recall from Section 1.9 that for all x in the domains of f and f^{-1} , inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Inverse Properties of Trigonometric Functions

If $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$, then

$$\cos(\arccos x) = x \quad \text{and} \quad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

Keep in mind that these inverse properties do not apply for arbitrary values of x and y . For instance,

$$\arcsin\left(\sin \frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2} \neq \frac{3\pi}{2}.$$

In other words, the property

$$\arcsin(\sin y) = y$$

is not valid for values of y outside the interval $[-\pi/2, \pi/2]$.

Example 5 Using Inverse Properties

If possible, find the exact value.

a. $\tan[\arctan(-5)]$ b. $\arcsin\left(\sin \frac{5\pi}{3}\right)$ c. $\cos(\cos^{-1} \pi)$

Solution

a. Because -5 lies in the domain of the arctan function, the inverse property applies, and you have

$$\tan[\arctan(-5)] = -5.$$

b. In this case, $5\pi/3$ does not lie within the range of the arcsine function, $-\pi/2 \leq y \leq \pi/2$. However, $5\pi/3$ is coterminal with

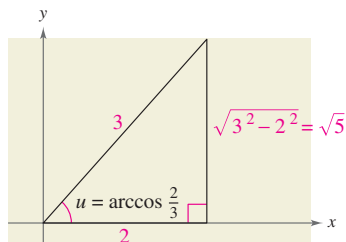
$$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

which does lie in the range of the arcsine function, and you have

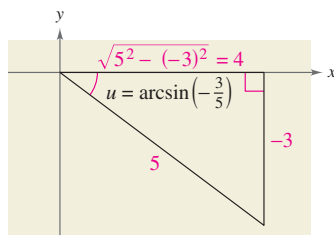
$$\arcsin\left(\sin \frac{5\pi}{3}\right) = \arcsin\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}.$$

c. The expression $\cos(\cos^{-1} \pi)$ is not defined because $\cos^{-1} \pi$ is not defined. Remember that the domain of the inverse cosine function is $[-1, 1]$.

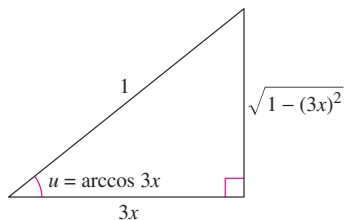
CheckPoint Now try Exercise 49.



Angle whose cosine is $\frac{2}{3}$
FIGURE 4.75



Angle whose sine is $-\frac{3}{5}$
FIGURE 4.76



Angle whose cosine is $3x$
FIGURE 4.77

Example 6 shows how to use right triangles to find exact values of compositions of inverse functions. Then, Example 7 shows how to use right triangles to convert a trigonometric expression into an algebraic expression. This conversion technique is used frequently in calculus.

Example 6 Evaluating Compositions of Functions

Find the exact value.

a. $\tan\left(\arccos\frac{2}{3}\right)$ b. $\cos\left[\arcsin\left(-\frac{3}{5}\right)\right]$

Solution

a. If you let $u = \arccos\frac{2}{3}$, then $\cos u = \frac{2}{3}$. Because $\cos u$ is positive, u is a *first*-quadrant angle. You can sketch and label angle u as shown in Figure 4.75. Consequently,

$$\tan\left(\arccos\frac{2}{3}\right) = \tan u = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{5}}{2}.$$

b. If you let $u = \arcsin\left(-\frac{3}{5}\right)$, then $\sin u = -\frac{3}{5}$. Because $\sin u$ is negative, u is a *fourth*-quadrant angle. You can sketch and label angle u as shown in Figure 4.76. Consequently,

$$\cos\left[\arcsin\left(-\frac{3}{5}\right)\right] = \cos u = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}.$$

CHECKPOINT Now try Exercise 57.

Example 7 Some Problems from Calculus

Write each of the following as an algebraic expression in x .

a. $\sin(\arccos 3x)$, $0 \leq x \leq \frac{1}{3}$ b. $\cot(\arccos 3x)$, $0 \leq x < \frac{1}{3}$

Solution

If you let $u = \arccos 3x$, then $\cos u = 3x$, where $-1 \leq 3x \leq 1$. Because

$$\cos u = \frac{\text{adj}}{\text{hyp}} = \frac{3x}{1}$$

you can sketch a right triangle with acute angle u , as shown in Figure 4.77. From this triangle, you can easily convert each expression to algebraic form.

a. $\sin(\arccos 3x) = \sin u = \frac{\text{opp}}{\text{hyp}} = \sqrt{1 - 9x^2}$, $0 \leq x \leq \frac{1}{3}$

b. $\cot(\arccos 3x) = \cot u = \frac{\text{adj}}{\text{opp}} = \frac{3x}{\sqrt{1 - 9x^2}}$, $0 \leq x < \frac{1}{3}$

CHECKPOINT Now try Exercise 67.

In Example 7, similar arguments can be made for x -values lying in the interval $[-\frac{1}{3}, 0]$.