# 4.4 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

# What You Should Learn

- Evaluate trigonometric functions of any angle.
- Find reference angles.
- Evaluate trigonometric functions of real numbers.

# Introduction

# Introduction

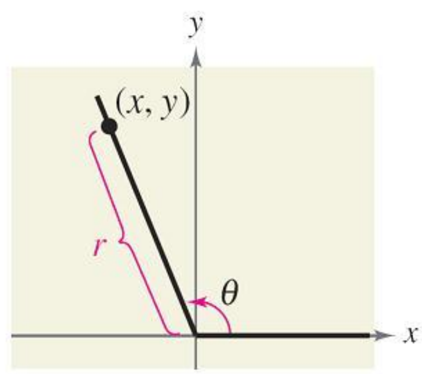
### Definitions of Trigonometric Functions of Any Angle

Let $\theta$ be an angle in standard position with $(x, y)$ a point on the terminal side of $\theta$ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x},\quad x \neq 0 \qquad \cot\theta = \frac{x}{y},\quad y \neq 0$$

$$\sec\theta = \frac{r}{x},\quad x \neq 0 \qquad \csc\theta = \frac{r}{y},\quad y \neq 0$$



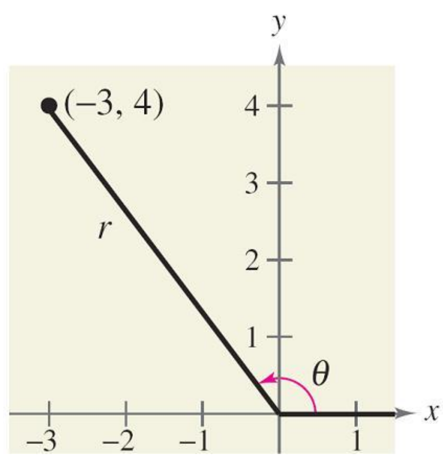
## Example 1 – *Evaluating Trigonometric Functions*

Let (–3, 4) be a point on the terminal side of $\theta$. Find the sine, cosine, and tangent of $\theta$.

Solution:

$x = -3$, $y = 4$,

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-3)^2 + 4^2}$$
$$= \sqrt{25}$$
$$= 5.$$



# Example 1 – *Solution*

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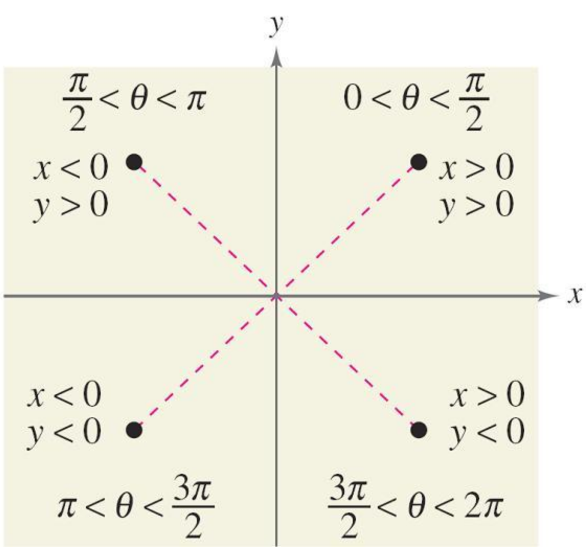
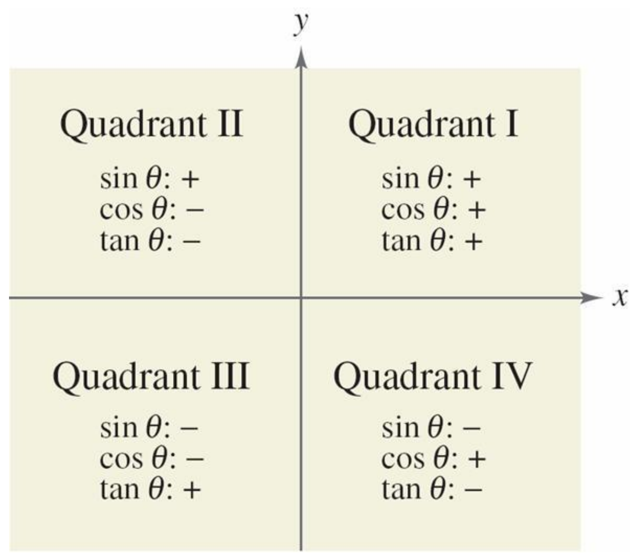
$$\sin\,\theta = \frac{y}{r} = \frac{4}{5}$$

$$\cos\,\theta = \frac{x}{r} = -\frac{3}{5}$$

$$\tan\,\theta = \frac{y}{x} = -\frac{4}{3}$$

# Introduction



# Reference Angles

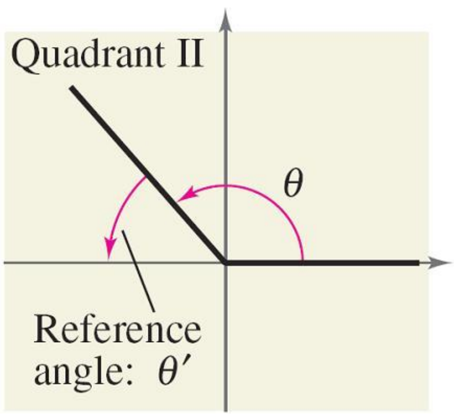
# Reference Angles

**Definition of Reference Angle**

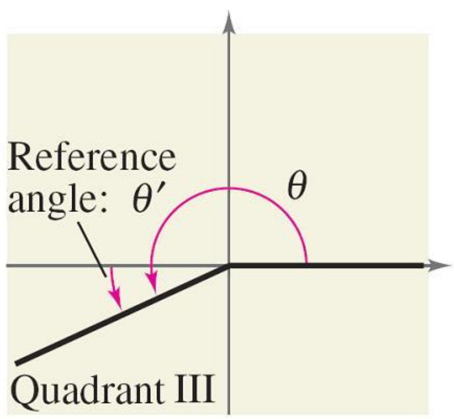
Let $\theta$ be an angle in standard position. Its **reference angle** is the acute angle $\theta'$ formed by the terminal side of $\theta$ and the horizontal axis.

# Reference Angles

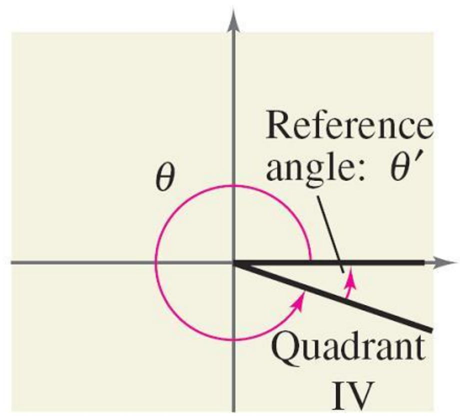
The reference angles for $\theta$ in Quadrants II, III, and IV.




$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)




$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)




$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

# Example 4 – *Finding Reference Angles*

Find the reference angle $\theta'$.

**a.** $\theta = 300°$

**b.** $\theta = 2.3$

**c.** $\theta = -135°$

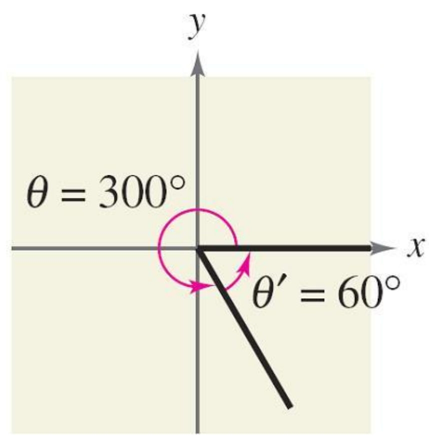
# Example 4(a) – *Solution*

Because 300° lies in Quadrant IV, the angle it makes with the $x$-axis is

$$\theta' = 360° - 300°$$

$$= 60°.$$

Degrees



# Example 4(b) – *Solution*

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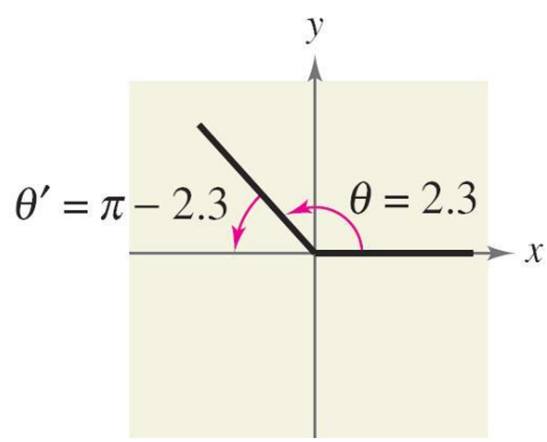
Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

$$\theta' = \pi - 2.3$$

$$\approx 0.8416.$$

Radians



# Example 4(c) – *Solution*

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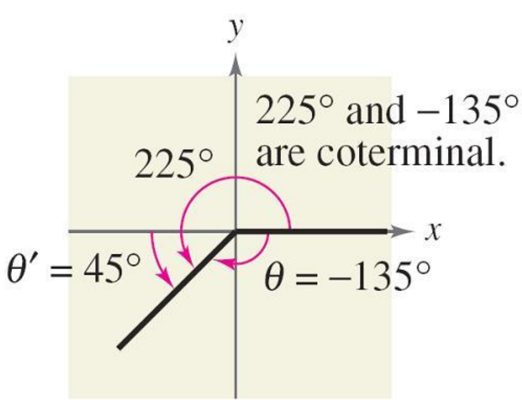
First, determine that –135° is coterminal with 225°, which lies in Quadrant III. So, the reference angle is

$$\theta' = 225° - 180°$$

$$= 45°.$$

Degrees



# Trigonometric Functions of Real Numbers

# Trigonometric Functions of Real Numbers

By definition, you know that

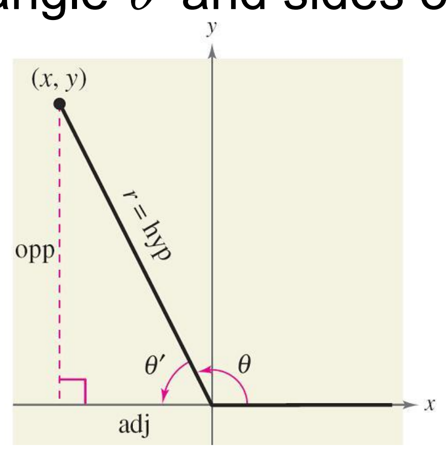
$$\sin\theta = \frac{y}{r} \qquad \text{and} \qquad \tan\theta = \frac{y}{x}.$$

For the right triangle with acute angle $\theta'$ and sides of lengths $|x|$ and $|y|$, you have

$$\sin\theta' = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r}$$

and

$$\tan\theta' = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}.$$




opp = $|y|$, adj = $|x|$

# Trigonometric Functions of Real Numbers

## Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle $\theta$:

1. Determine the function value for the associated reference angle $\theta'$.
2. Depending on the quadrant in which $\theta$ lies, affix the appropriate sign to the function value.

| $\theta$ (degrees) | $0°$ | $30°$ | $45°$ | $60°$ | $90°$ | $180°$ | $270°$ |
|---|---|---|---|---|---|---|---|
| $\theta$ (radians) | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
| $\sin\theta$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $0$ | $-1$ |
| $\cos\theta$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-1$ | $0$ |
| $\tan\theta$ | $0$ | $\frac{\sqrt{3}}{3}$ | $1$ | $\sqrt{3}$ | Undef. | $0$ | Undef. |

# Example 5 – *Using Reference Angles*

Evaluate each trigonometric function.

**a.** cos $\frac{4\pi}{3}$

**b.** tan(–210°)

**c.** csc $\frac{11\pi}{4}$

# Example 5(a) – *Solution*

cont'd

Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is

$$\theta' = \frac{4\pi}{3} - \pi$$

$$= \frac{\pi}{3}$$

as shown in Figure 4.44.

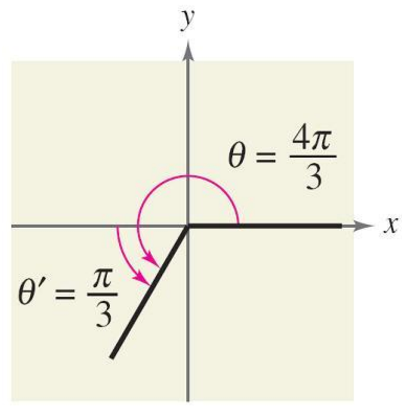



**Figure 4.44**

Moreover, the cosine is negative in Quadrant III, so

$$\cos\frac{4\pi}{3} = (-)\cos\frac{\pi}{3}$$

$$= -\frac{1}{2}.$$

# Example 5(b) – *Solution*

cont'd

Because –210° + 360° = 150°, it follows that –210° is coterminal with the second-quadrant angle 150°.

So, the reference angle is $\theta'$ = 180° – 150° = 30°, as shown in Figure 4.45.

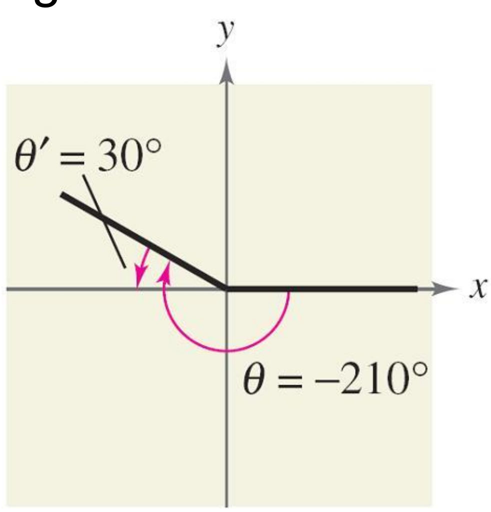



**Figure 4.45**

# Example 5(b) – *Solution*

cont'd

Finally, because the tangent is negative in Quadrant II, you have

$$\tan(-210°) = (-)\tan 30°$$

$$= -\frac{\sqrt{3}}{3}.$$

# Example 5(c) – *Solution*

cont'd

Because $(11\pi/4) - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$.

So, the reference angle is $\theta' = \pi - (3\pi/4) = \pi/4$, as shown in Figure 4.46.

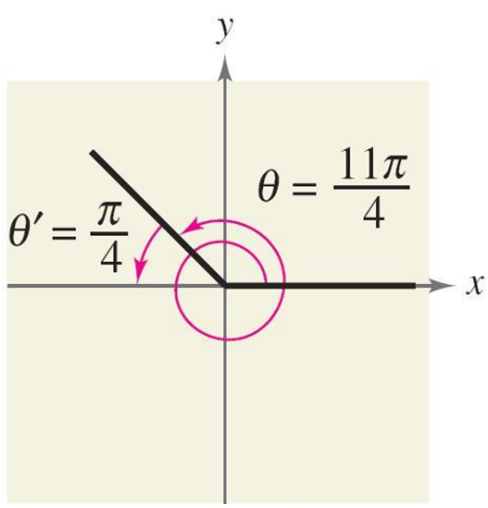



**Figure 4.46**

# Example 5(c) – *Solution*

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Because the cosecant is positive in Quadrant II, you have

$$\csc \frac{11\pi}{4} = (+) \csc \frac{\pi}{4} = \frac{1}{\sin(\pi/4)} = \sqrt{2}.$$