4.2 TRIGONOMETRIC FUNCTIONS: THE UNIT CIRCLE

What You Should Learn

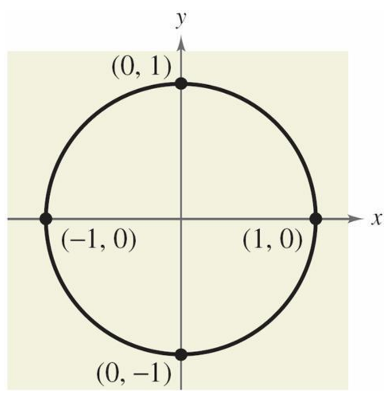
- Identify a unit circle and describe its relationship to real numbers.
- Evaluate trigonometric functions using the unit circle.
- Use the domain and period to evaluate sine and cosine functions.
- Use a calculator to evaluate trigonometric functions.

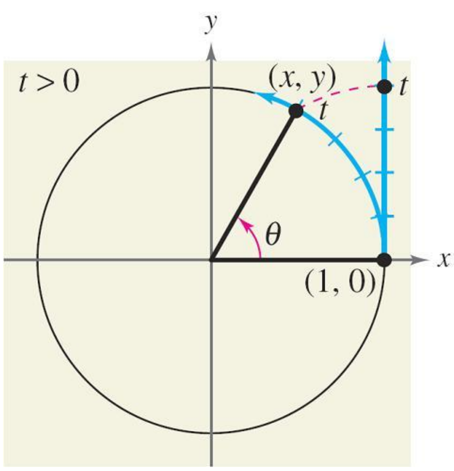
The Unit Circle

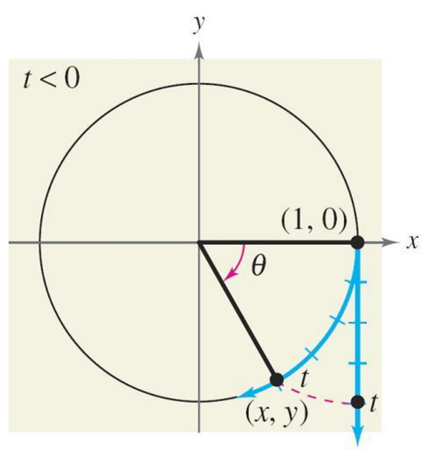
The Unit Circle

The **unit circle**

$$x^2 + y^2 = 1$$







The Unit Circle

Each real number t also corresponds to a central angle θ (in standard position) whose radian measure is t.

The real number t is the (directional) length of the arc intercepted by the angle θ, given in radians.

The Trigonometric Functions

The Trigonometric Functions

Definitions of Trigonometric Functions

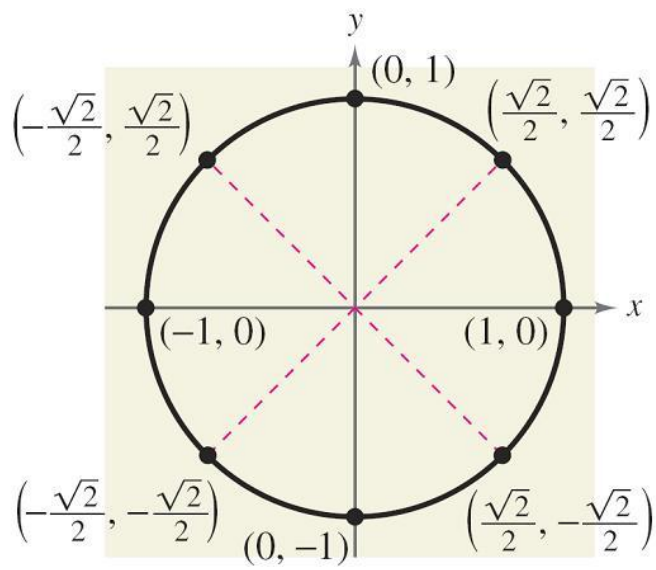
Let t be a real number and let (x, y) be the point on the unit circle corresponding to t.

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}, \quad x \neq 0$$

$$\csc t = \frac{1}{y}, \quad y \neq 0 \qquad \sec t = \frac{1}{x}, \quad x \neq 0 \qquad \cot t = \frac{x}{y}, \quad y \neq 0$$

The Trigonometric Functions

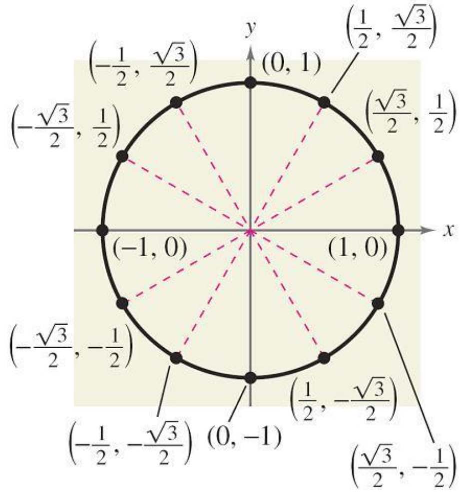
The unit circle has been divided into eight equal arcs, corresponding to *t*-values of $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$, and 2π.



The Trigonometric Functions

The unit circle has been divided into 12 equal arcs, corresponding to *t*-values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$



Example 1 – *Evaluating Trigonometric Functions*

Evaluate the six trigonometric functions at each real number.

a. $t = \frac{\pi}{6}$ **b.** $t = \frac{5\pi}{4}$ **c.** $t = 0$ **d.** $t = \pi$

Solution:

a. $t = \frac{\pi}{6}$ corresponds to the point $(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

$$\sin \frac{\pi}{6} = y = \frac{1}{2}$$

$$\cos \frac{\pi}{6} = x = \frac{\sqrt{3}}{2}$$

Example 1 – *Solution*

cont'd

$$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc\frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

$$\sec\frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot\frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Example 1 – *Solution*

cont'd

b. $t = \dfrac{5\pi}{4}$ corresponds to the point (x, y) = $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.

$$\sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2}$$

$$\cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2}$$

$$\tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

$$\csc\frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

Example 1 – *Solution*

cont'd

$$\sec \frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cot \frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

Example 1 – *Solution*

cont'd

c. $t = 0$ corresponds to the point $(x, y) = (1, 0)$.

$$\sin 0 = y = 0$$

$$\cos 0 = x = 1$$

$$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$$

Example 1 – *Solution*

cont'd

$\csc 0 = \frac{1}{y}$ is undefined.

$$\sec 0 = \frac{1}{x} = \frac{1}{1} = 1$$

$\cot 0 = \frac{x}{y}$ is undefined.

Example 1 – *Solution*

cont'd

d. $t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

$$\sin \pi = y = 0$$

$$\cos \pi = x = -1$$

$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$$

Example 1 – *Solution*

cont'd

$\csc \pi = \frac{1}{y}$ is undefined.

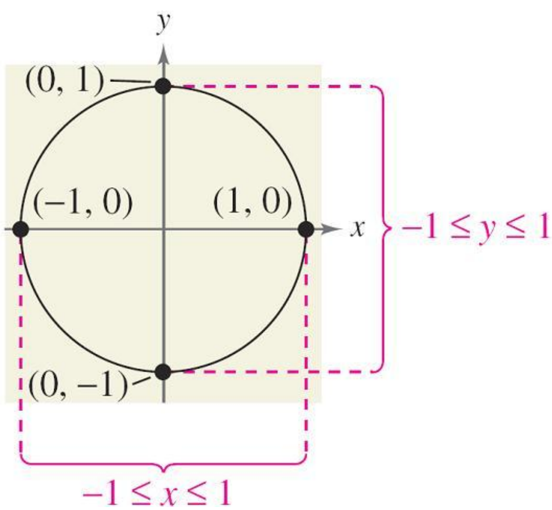
$$\sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$\cot \pi = \frac{x}{y}$ is undefined.

Domain and Period of Sine and Cosine

Domain and Period of Sine and Cosine

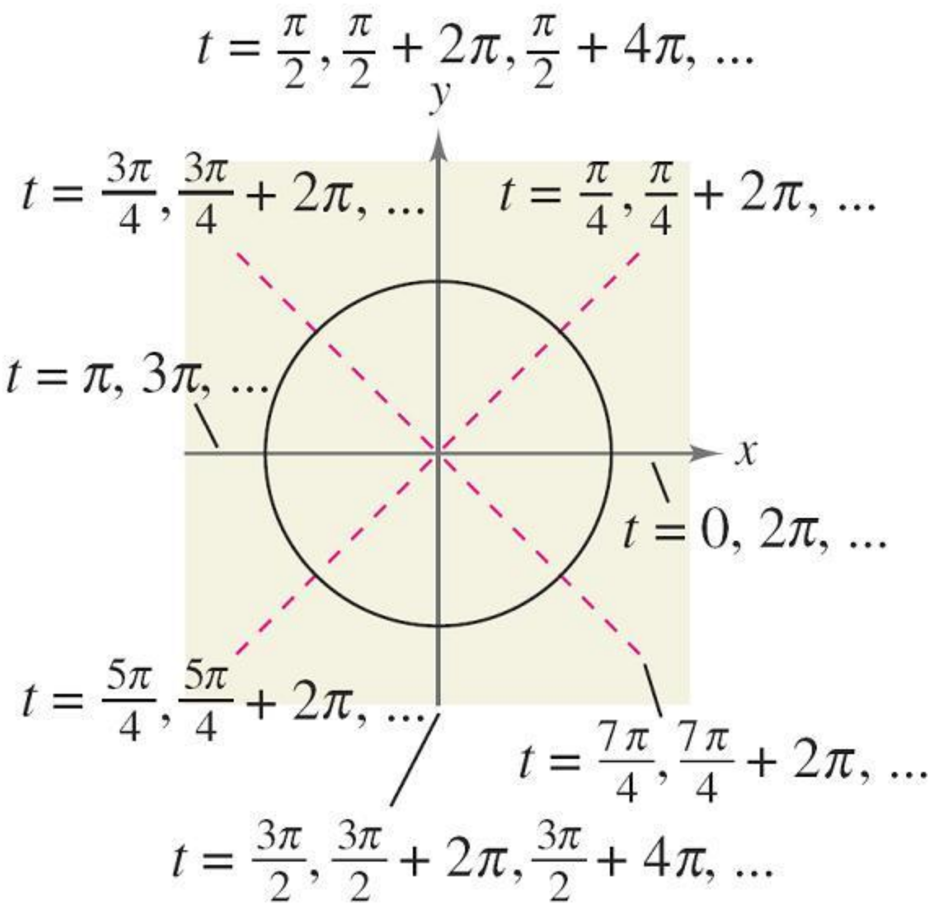
The *domain* of the sine and cosine functions is the set of all real numbers.



$$-1 \leq y \leq 1$$
$$-1 \leq \text{Sin } t \leq 1$$

$$-1 \leq x \leq 1$$
$$-1 \leq \cos t \leq 1$$

Domain and Period of Sine and Cosine



$\sin(t + 2\pi n) = \sin t$

$\cos(t + 2\pi n) = \cos t$

Domain and Period of Sine and Cosine

Definition of Periodic Function

A function f is **periodic** if there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f. The smallest number c for which f is periodic is called the **period** of f.

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$$\sin(-t) = -\sin t \qquad \csc(-t) = -\csc t$$

$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

Example 3 – *Using the Period to Evaluate the Sine and Cosine*

a. Because $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, you have

$$\sin\frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right)$$

$$= \sin\frac{\pi}{6}$$

$$= \frac{1}{2}.$$

b. Because $-\frac{7\pi}{2} = -4\pi + \frac{\pi}{2}$, you have

$$\cos\left(-\frac{7\pi}{2}\right) = \cos\left(-4\pi + \frac{\pi}{2}\right)$$

Example 3 – *Using the Period to Evaluate the Sine and Cosine*

cont'd

$$= \cos \frac{\pi}{2}$$

$$= 0.$$

c. For $\sin t = \frac{4}{5}$, $\sin(-t) = -\frac{4}{5}$ because the sine function is odd.