Analytic Trigonometry

5

More…

Find all solutions of the equations in the interval $[0,2\pi)$

1. $\cos(x - \frac{\pi}{2}) + \sin^2 x = 0$
2. $\tan(x + \pi) + 2\sin(x + \pi) = 0$

Write the trigonometric expression as an algebraic expression:
$\sin(\arctan 2x - \arccos x)$

5.5 MULTIPLE-ANGLE AND PRODUCT-TO-SUM FORMULAS

Multiple-Angle Formulas

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\begin{aligned} \cos 2u &= \cos^2 u - \sin^2 u \\ &= 2 \cos^2 u - 1 \\ &= 1 - 2 \sin^2 u \end{aligned}$$

Example 1 – *Solving a Multiple-Angle Equation*

Solve $2 \cos x + \sin 2x = 0$.

Solution:

$$2 \cos x + \sin 2x = 0$$

$$2 \cos x + 2 \sin x \cos x = 0$$

$$2 \cos x(1 + \sin x) = 0$$

$$2 \cos x = 0 \quad \text{and} \quad 1 + \sin x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{3\pi}{2} \qquad x = \frac{\pi}{2} + 2n\pi \qquad x = \frac{3\pi}{2} + 2n\pi$$

Example 3: Evaluating Functions Involving Double Angles

Use the $\cos\theta = \frac{5}{13}, \frac{3\pi}{2} < \theta < 2\pi$ to find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$

Solution:

$$\cos\theta = \frac{5}{13}, \frac{3\pi}{2} < \theta < 2\pi \Rightarrow \sin\theta = -\sqrt{1-\left(\frac{5}{13}\right)^2} = -\frac{2}{13}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{2}{13}\right)\left(\frac{5}{13}\right) = -\frac{120}{169}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{5}{13}\right)^2 - 1 = -\frac{119}{169}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{120}{119}$$

Power-Reducing Formulas

$$\cos 2u = 2\cos^2 u - 1 = 1 - 2\sin^2 2u = \cos^2 2u - \sin^2 2u$$

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2} \qquad \tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Example 5 – *Reducing a Power*

Rewrite $\sin^4 x$ as a sum of first powers of the cosines of multiple angles.

Solution:

$$\sin^4 x = (\sin^2 x)^2 \quad = \left(\frac{1 - \cos 2x}{2}\right)^2$$

$$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right)$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$$

$$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

Half-Angle Formulas

Half-Angle Formulas

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}}$$

$$\tan \frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The signs of $\sin \frac{u}{2}$ and $\cos \frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2} \qquad \cos^2 u = \frac{1 + \cos 2u}{2} \qquad \tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Example 6 – *Using a Half-Angle Formula*

Find the exact value of sin 105°.

Solution:

$$\sin 105^\circ = \sqrt{\frac{1 - \cos 210^\circ}{2}}$$

$$= \sqrt{\frac{1 - (-\cos 30^\circ)}{2}}$$

$$= \sqrt{\frac{1 + \left(\sqrt{3}/2\right)}{2}}$$

$$= \frac{\sqrt{2 + \sqrt{3}}}{2}$$

Product-to-Sum Formulas

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$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

Example 8 – *Writing Products as Sums*

Rewrite the product $\cos 5x \sin 4x$ as a sum or difference.

Solution:

$$\cos 5x \sin 4x = (½)\,[\sin(5x + 4x) - \sin(5x - 4x)]$$

$$= (½)\,(\sin 9x - \sin x)$$

Product-to-Sum Formulas

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Example 9

Find the exact value of $\cos 195° + \cos 105°$

Solution:

$\cos 195° + \cos 105°$

$$= 2\cos\left(\frac{195° + 105°}{2}\right)\cos\left(\frac{195° - 105°}{2}\right)$$

$$= 2\cos 150° \cos 45°$$

$$= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{2}$$

Example 11

Verify the identity $\frac{\sin 3x - \sin x}{\cos x + \cos 3x} = \tan x$

Proof :

$$\frac{\sin 3x - \sin x}{\cos x + \cos 3x} = \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{2\cos\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}$$

$$= \frac{2\cos 2x \sin x}{2\cos 2x \cos(-x)}$$

$$= \frac{\sin x}{\cos x} = \tan x$$

Application

Example 12 – *Projectile Motion*

Ignoring air resistance, the range of a projectile fired at an angle θ with the horizontal and with an initial velocity of v_0 feet per second is given by

$$r = \frac{1}{16} v_0^2 \sin \theta \cos \theta$$

where r is the horizontal distance (in feet) that the projectile will travel.

Example 12 – *Projectile Motion*

cont'd

A place kicker for a football team can kick a football from ground level with an initial velocity of 80 feet per second (see Figure 5.18).

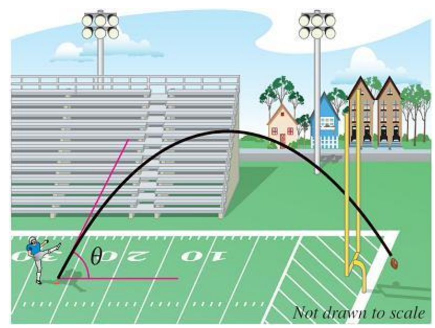


Figure 5.18

a. Write the projectile motion model in a simpler form.

b. At what angle must the player kick the football so that the football travels 200 feet?

c. For what angle is the horizontal distance the football travels a maximum?

Example 12 – *Solution*

a. You can use a double-angle formula to rewrite the projectile motion model as

$$r = \frac{1}{32} v_0^2 (2 \sin \theta \cos \theta)$$ Rewrite original projectile motion model.

$$= \frac{1}{32} v_0^2 \sin 2\theta.$$ Rewrite model using a double-angle formula.

b. $$r = \frac{1}{32} v_0^2 \sin 2\theta$$ Write projectile motion model.

$$200 = \frac{1}{32} (80)^2 \sin 2\theta$$ Substitute 200 for r and 80 for v_0.

Example 12 – *Solution*

cont'd

$200 = 200 \sin 2\theta$ Simplify.

$1 = \sin 2\theta$ Divide each side by 200.

You know that $2\theta = \pi/2$, so dividing this result by 2 produces $\theta = \pi/4$.

Because $\pi/4 = 45°$, you can conclude that the player must kick the football at an angle of 45° so that the football will travel 200 feet.

Example 12 – *Solution*

cont'd

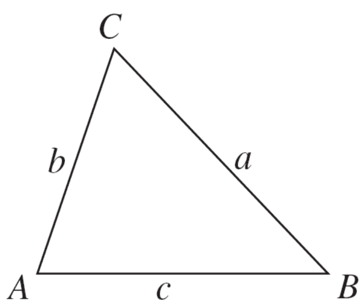
c. From the model $r = 200 \sin 2\theta$ you can see that the amplitude is 200. So the maximum range is $r = 200$ feet.

From part (b), you know that this corresponds to an angle of 45°.

Therefore, kicking the football at an angle of 45° will produce a maximum horizontal distance of 200 feet.

Introduction

Oblique triangles—triangles that have no right angles.



Law of Sine

1. Two angles and any side (AAS or ASA)

2. Two sides and an angle opposite one of them (SSA)

Law of Cosine

3. Three sides (SSS)

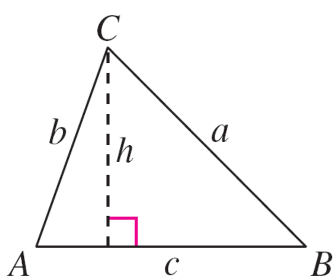
4. Two sides and their included angle (SAS)

Introduction

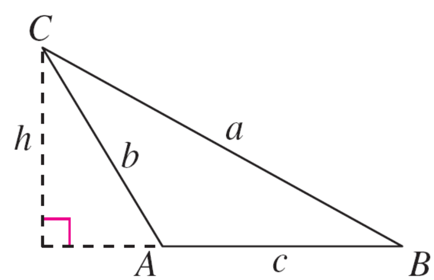
Law of Sines

If ABC is a triangle with sides a, b, and c, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



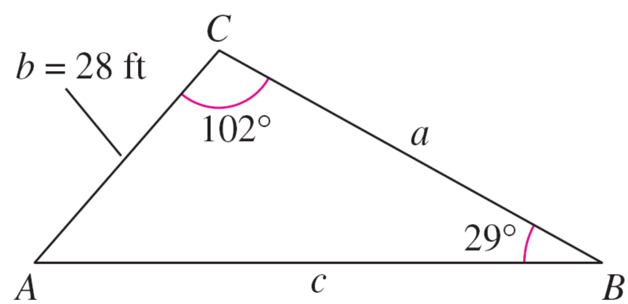
A is acute.



A is obtuse.

Example 1 – *Given Two Angles and One Side—AAS*

For the triangle in figure, $C = 120°$, $B = 29°$, and $b = 28$ feet. Find the remaining angle and sides.



Solution:

The third angle of the triangle is

$$A = 180° - B - C$$

$$= 180° - 29° - 102°$$

$$= 49°.$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a = \frac{b}{\sin B}(\sin A) = \frac{28}{\sin 29°}(\sin 49°) \approx 43.59 \text{ feet}$$

$$c = \frac{b}{\sin B}(\sin C) = \frac{28}{\sin 29°}(\sin 102°) \approx 56.49 \text{ feet.}$$

Example 1 – *Solution*

The third angle of the triangle is

$A = 180° - B - C$

$= 180° - 29° - 102°$

$= 49°.$

By the Law of Sines, you have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$$a = \frac{b}{\sin B}(\sin A) = \frac{28}{\sin 29°}(\sin 49°) \approx 43.59 \text{ feet}$$

$$c = \frac{b}{\sin B}(\sin C) = \frac{28}{\sin 29°}(\sin 102°) \approx 56.49 \text{ feet.}$$