The **bisection method** for approximating the real zeros of a continuous function is similar to the method used in Example 8. If you know that a zero exists in the closed interval $[a, b]$, the zero must lie in the interval $[a, (a+b)/2]$ or $[(a+b)/2, b]$. From the sign of $f([a+b]/2)$, you can determine which interval contains the zero. By repeatedly bisecting the interval, you can "close in" on the zero of the function.

TECHNOLOGY You can also use the *zoom* feature of a graphing utility to approximate the real zeros of a continuous function. By repeatedly zooming in on the point where the graph crosses the x-axis, and adjusting the x-axis scale, you can approximate the zero of the function to any desired accuracy. The zero of $x^3 + 2x - 1$ is approximately 0.453, as shown in Figure 1.38.

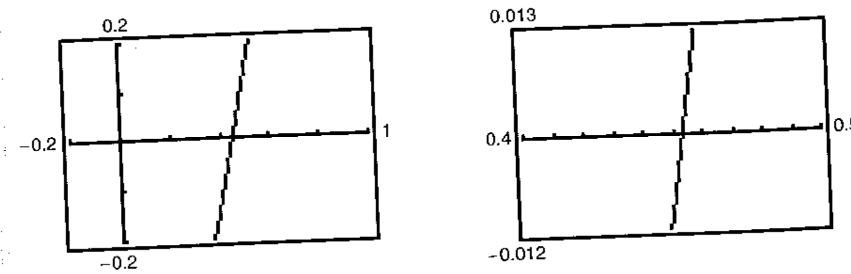
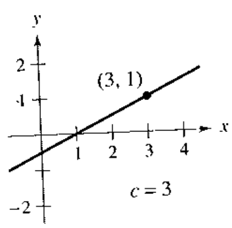


Figure 1.38 Zooming in on the zero of $f(x) = x^3 + 2x - 1$

EXERCISES FOR SECTION 1.4

In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim_{x\to c^+} f(x)$ (b) $\lim_{x\to c^-} f(x)$ (c) $\lim_{x\to c} f(x)$

1.


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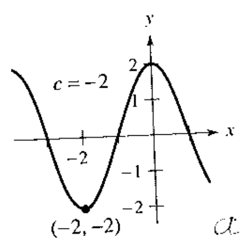

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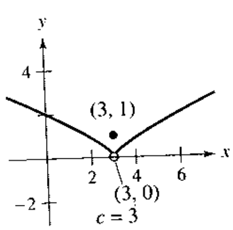

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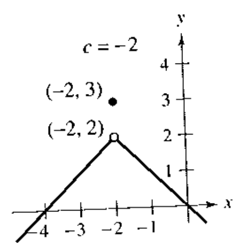

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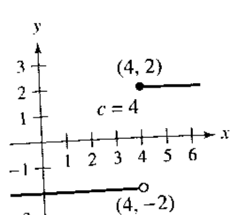
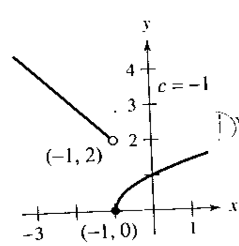


6.


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In Exercises 7–24, find the limit (if it exists). If it does not exist, explain why.

7. $\lim_{x\to 5^+} \dfrac{x-5}{x^2-25}$

8. $\lim_{x\to 2^-} \dfrac{2-x}{x^2-4}$

9. $\lim_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$

10. $\lim_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4}$

11. $\lim_{x\to 0^-} \dfrac{|x|}{x}$

12. $\lim_{x\to 2^+} \dfrac{|x-2|}{x-2}$

13. $\lim_{\Delta x\to 0^-} \dfrac{\dfrac{1}{x+\Delta x} - \dfrac{1}{x}}{\Delta x}$

14. $\lim_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2 + x + \Delta x - (x^2+x)}{\Delta x}$

15. $\lim_{x\to 3^-} f(x)$, where $f(x) = \begin{cases} \dfrac{x+2}{2}, & x \le 3 \\ \dfrac{12-2x}{3}, & x > 3 \end{cases}$

16. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} x^2 - 4x + 6, & x < 2 \\ -x^2 + 4x - 2, & x \ge 2 \end{cases}$

17. $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^3 + 1, & x < 1 \\ x + 1, & x \ge 1 \end{cases}$

18. $\lim_{x\to 1^+} f(x)$, where $f(x) = \begin{cases} x, & x \le 1 \\ 1 - x, & x > 1 \end{cases}$

19. $\lim_{x\to \pi} \cot x$

20. $\lim_{x\to \pi/2} \sec x$

21. $\lim_{x\to 4^-} (3[\![x]\!] - 5)$

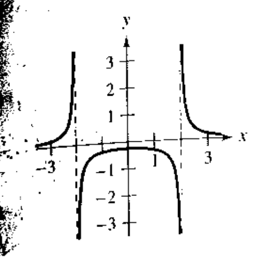
22. $\lim_{x\to 2^+} (2x - [\![x]\!])$

23. $\lim_{x\to 3} (2 - [\![-x]\!])$

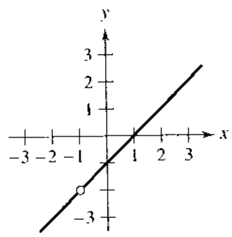
24. $\lim_{x\to 1} \left(1 - [\![-\frac{x}{2}]\!]\right)$

In Exercises 25–28, discuss the continuity of each function.

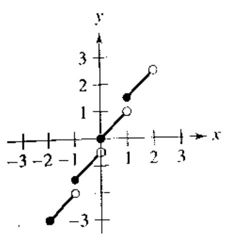
25. $f(x) = \dfrac{1}{x^2 - 4}$



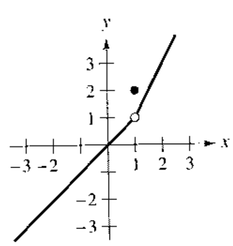
26. $f(x) = \dfrac{x^2 - 1}{x + 1}$



27. $f(x) = \frac{1}{2}[\![x]\!] + x$



28. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$



In Exercises 29–32, discuss the continuity of the function on the closed interval.

29. $g(x) = \sqrt{25 - x^2}$, $[-5, 5]$

30. $f(t) = 3 - \sqrt{9 - t^2}$, $[-3, 3]$

31. $f(x) = \begin{cases} 3 - x, & x \le 0 \\ 3 + \frac{1}{2}x, & x > 0 \end{cases}$, $[-1, 4]$

32. $g(x) = \dfrac{1}{x^2 - 4}$, $[-1, 2]$

In Exercises 33–54, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

33. $f(x) = x^2 - 2x + 1$

34. $f(x) = \dfrac{1}{x^2 + 1}$

35. $f(x) = 3x - \cos x$

36. $f(x) = \cos \dfrac{\pi x}{2}$

37. $f(x) = \dfrac{x}{x^2 - x}$

38. $f(x) = \dfrac{x}{x^2 - 1}$

39. $f(x) = \dfrac{x}{x^2 + 1}$

40. $f(x) = \dfrac{x - 3}{x^2 - 9}$

41. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$

42. $f(x) = \dfrac{x - 1}{x^2 + x - 2}$

43. $f(x) = \dfrac{|x + 2|}{x + 2}$

44. $f(x) = \dfrac{|x - 3|}{x - 3}$

45. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

46. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

47. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

48. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

49. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

50. $f(x) = \begin{cases} \csc \frac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

51. $f(x) = \csc 2x$

52. $f(x) = \tan \dfrac{\pi x}{2}$

53. $f(x) = [\![x - 1]\!]$

54. $f(x) = 3 - [\![x]\!]$

In Exercises 55 and 56, use a graphing utility to graph the function. From the graph, estimate

$$\lim_{x \to 0^+} f(x) \quad \text{and} \quad \lim_{x \to 0^-} f(x).$$

Is the function continuous on the entire real line? Explain.

55. $f(x) = \dfrac{|x^2 - 4|x}{x + 2}$

56. $f(x) = \dfrac{|x^2 + 4x|(x + 2)}{x + 4}$

In Exercises 57–60, find the constants a and b such that the function is continuous on the entire real line.

57. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

58. $g(x) = \begin{cases} \frac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

59. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

60. $g(x) = \begin{cases} \frac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

In Exercises 61–64, discuss the continuity of the composite function $h(x) = f(g(x))$.

61. $f(x) = x^2$
$g(x) = x - 1$

62. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

63. $f(x) = \dfrac{1}{x - 6}$
$g(x) = x^2 + 5$

64. $f(x) = \sin x$
$g(x) = x^2$

In Exercises 65–68, use a graphing utility to graph the function. Use the graph to determine any x-values at which the function is not continuous.

65. $f(x) = [\![x]\!] - x$

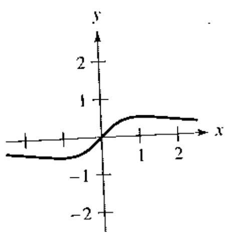
66. $h(x) = \dfrac{1}{x^2 - x - 2}$

67. $g(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

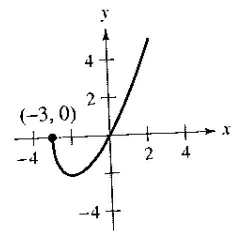
68. $f(x) = \begin{cases} \frac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

In Exercises 69–72, describe the interval(s) on which the function is continuous.

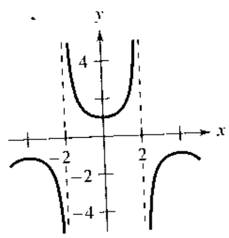
69. $f(x) = \dfrac{x}{x^2 + 1}$



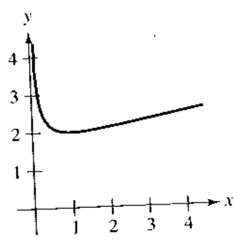
70. $f(x) = x\sqrt{x + 3}$



71. $f(x) = \sec \dfrac{\pi x}{4}$



72. $f(x) = \dfrac{x + 1}{\sqrt{x}}$



Writing **In Exercises 73 and 74, use a graphing utility to graph the function on the interval [−4, 4]. Does the graph of the function appear continuous on this interval? Is the function continuous on [−4, 4]? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

73. $f(x) = \dfrac{\sin x}{x}$

74. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 75–78, explain why the function has a zero in the specified interval.**

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3, \quad [1, 2]$

76. $f(x) = x^3 + 3x - 2, \quad [0, 1]$

77. $f(x) = x^2 - 2 - \cos x, \quad [0, \pi]$

78. $f(x) = -\dfrac{4}{x} + \tan\left(\dfrac{\pi x}{8}\right), \quad [1, 3]$

In Exercises 79–82, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval [0, 1]. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the root-finding capabilities of the graphing utility to approximate the zero accurate to four decimal places.

79. $f(x) = x^3 + x - 1$

80. $f(x) = x^3 + 3x - 2$

81. $g(t) = 2 \cos t - 3t$

82. $h(\theta) = 1 + \theta - 3 \tan \theta$

In Exercises 83–86, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of c guaranteed by the theorem.

83. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

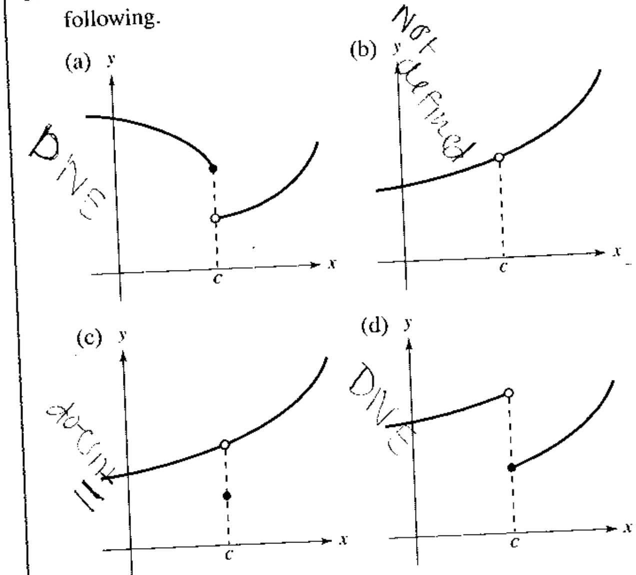
84. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

85. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

86. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

Getting at the Concept

87. State how continuity is destroyed at $x = c$ for each of the following.



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88. Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following.
   (a) A function with a nonremovable discontinuity at $x = 2$.
   (b) A function with a removable discontinuity at $x = -2$.
   (c) A function that has both of the characteristics described in parts (a) and (b).

89. Sketch the graph of any function f such that
$$\lim_{x \to 3^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 3^-} f(x) = 0.$$
Is the function continuous at $x = 3$? Explain.

90. If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

91. ***Think About It*** Describe how the functions $f(x) = 3 + [\![x]\!]$ and $g(x) = 3 - [\![-x]\!]$ differ.