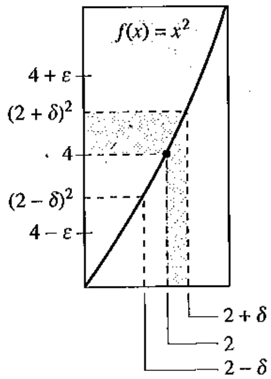


The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.15

Example 8 Using the ε-δ Definition of a Limit

Use the ε-δ definition of a limit to prove that

$$\lim_{x\to 2} x^2 = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon \text{ when } 0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in the interval $(1, 3)$, you know that $|x + 2| < 5$. So, letting δ be the minimum of $\varepsilon/5$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

as shown in Figure 1.15.

Throughout this chapter you will use the ε-δ definition of a limit primarily to prove theorems about limits and to establish the existence or nonexistence of particular types of limits. For *finding* limits, you will learn techniques that are easier to use than the ε-δ definition of a limit.

EXERCISES FOR SECTION 1.2

In Exercises 1–8, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\displaystyle\lim_{x\to 2} \frac{x - 2}{x^2 - x - 2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

2. $\displaystyle\lim_{x\to 2} \frac{x - 2}{x^2 - 4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x}$

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

4. $\displaystyle\lim_{x\to -3} \frac{\sqrt{1 - x} - 2}{x + 3}$

x	−3.1	−3.01	−3.001	−2.999	−2.99	−2.9
$f(x)$						

5. $\displaystyle\lim_{x\to 3} \frac{[1/(x + 1)] - (1/4)}{x - 3}$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

6. $\displaystyle\lim_{x\to 4} \frac{[x/(x + 1)] - (4/5)}{x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

7. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

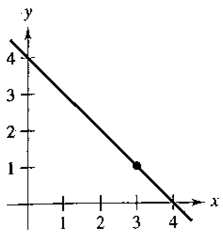
x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

8. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x}$

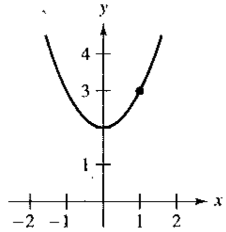
x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

In Exercises 9–18, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

9. $\lim_{x \to 3} (4 - x)$

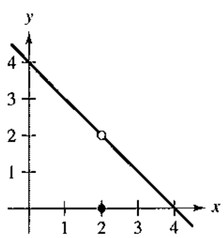


10. $\lim_{x \to 1} (x^2 + 2)$



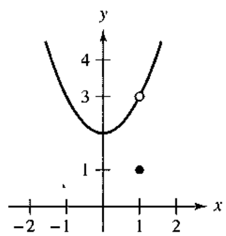
11. $\lim_{x \to 2} f(x)$

$f(x) = \begin{cases} 4 - x, & x \neq 2 \\ 0, & x = 2 \end{cases}$

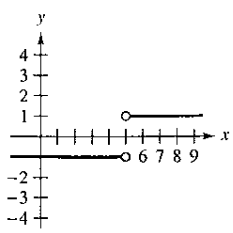


12. $\lim_{x \to 1} f(x)$

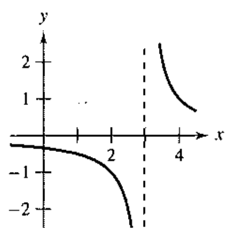
$f(x) = \begin{cases} x^2 + 2, & x \neq 1 \\ 1, & x = 1 \end{cases}$



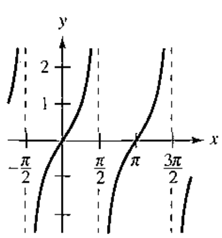
13. $\lim_{x \to 5} \frac{|x - 5|}{x - 5}$



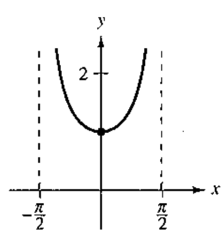
14. $\lim_{x \to 3} \frac{1}{x - 3}$



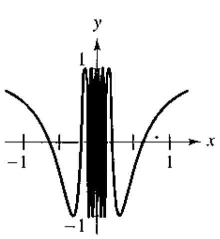
15. $\lim_{x \to \pi/2} \tan x$



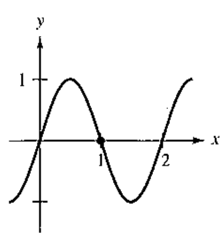
16. $\lim_{x \to 0} \sec x$



17. $\lim_{x \to 0} \cos \frac{1}{x}$



18. $\lim_{x \to 1} \sin \pi x$



19. *Modeling Data* The cost of a telephone call between two cities is \$0.75 for the first minute and \$0.50 for each additional minute. A formula for the cost is given by

$C(t) = 0.75 - 0.50[\![-(t - 1)]\!]$

where t is the time in minutes.

(*Note:* $[\![x]\!]$ = greatest integer n such that $n \leq x$. For example, $[\![3.2]\!] = 3$ and $[\![-1.6]\!] = -2$.)

(a) Use a graphing utility to graph the cost function for $0 < t \leq 5$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find

$\lim_{t \to 3.5} C(t)$.

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

20. Repeat Exercise 19 if $C(t) = 0.35 - 0.12[\![-(t - 1)]\!]$.

21. The graph of $f(x) = 2 - 1/x$ is shown in the figure. Find δ such that if $0 < |x - 1| < \delta$ then $|f(x) - 1| < 0.1$.

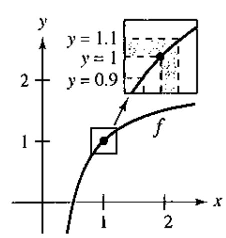


Figure for 21

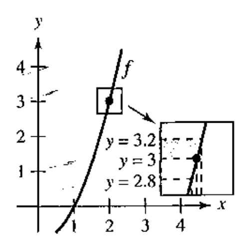


Figure for 22

22. The graph of $f(x) = x^2 - 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 3| < 0.2$.

In Exercises 23–26, find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

23. $\lim_{x \to 2} (3x + 2)$

24. $\lim_{x \to 4} \left(4 - \frac{x}{2}\right)$

25. $\lim_{x \to 2} (x^2 - 3)$

26. $\lim_{x \to 5} (x^2 + 4)$